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MATHEMATICAL ASPECTS OF THE THREE-BODY PROBLEM IN THE QUANTUM SCATTERING THEORY

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INTRODUCTION

Quantum mechanics is known to raise profound problems in the spectral theory of self-adjoint operators in Hilbert space. In particular, interesting mathematical problems associated with the perturbation theory of continuous spectra arise in the quantum theory of scattering.

Let us have a closer look at this situation. The energy operator of N particles is represented by a differential operator in the space of functions of the coordinates of all particles

$$\mathbf{H}_N = -\sum_{i=1}^{N} \frac{1}{2m_i} \nabla_i^2 + \sum_{\substack{i,j=1\\ i>j}}^{N} v_{ij}(x_i - x_j), \tag{0.1}$$

where x_i is the radius vector of the i-th particle, ∇_i^2 the Laplace operator for this particle, and m_i its mass. The functions $v_{ij}(x_i - x_j)$ represent the interaction between the i-th and the j-th particles. In scattering theory these functions fall off to zero as the inter-particle distance increases.

The differential operator $\mathbf{H}_N$ gives rise to a self-adjoint operator in the Hilbert space of square-integrable functions of the variables $x_1, \dots, x_N$. The first term of (0.1) corresponds to an operator with a purely continuous spectrum. It is reasonable to investigate how far is this continuous spectrum modified by the second term.

Specifically, it is an interesting mathematical problem to determine a complete set of eigenfunctions of $\mathbf{H}_N$, and prove that an arbitrary function can be expanded in them. Scattering theory not only poses the problem, but also suggests its solution: the so-called stationary solutions of the scattering problem are a natural choice for the set of such eigenfunctions.

In the stationary formulation of the scattering problem it is required to find the solutions of the time-independent Schrödinger equation

$$\mathbf{H}_N \psi(x_1, \dots, x_N) = E\psi(x_1, \dots, x_N), \tag{0.2}$$

which satisfy certain asymptotic conditions at infinity in configuration space. It should be emphasized that these conditions have been rigorously formulated only for a two-body system. In this case it is required to find for the Schrödinger equation

$$\mathbf{h}\psi(x, k) = -\frac{1}{2m}\nabla^2\psi(x, k) + v(x)\psi(x, k) = \frac{k^2}{2m}\psi(x, k),$$

solutions of the form

$$\psi(x, k) = \exp\{i(k, x)\} + w(x, k),$$

where k is a vector which in some way specifies the relative momentum of the particles, and $w(x, k)$ is the scattered wave, which satisfies the radiation condition for $|x| \to \infty$

$$w(x, k) = O\left(\frac{1}{|x|}\right); \quad \left(\frac{\partial}{\partial |x|} - i|k|\right) w(x, k) = o\left(\frac{1}{|x|}\right),$$

or, more accurately, behaves asymptotically for $|x| \to \infty$ as

$$w(x, k) = f(n, k)\frac{\exp\{i|k||x|\}}{|x|} + o\left(\frac{1}{|x|}\right); \quad n = \frac{x}{|x|}.$$

This problem was studied in detail by A. Ya. Povzner /1, 2/, who proved the existence of solutions $\psi(x, k)$ under certain conditions imposed on the potential $v(x)$, and showed that these functions constitute together with the eigenfunctions of the discrete spectrum of the operator $\mathbf{h}$ a complete system, which allows the expansion of any function in a generalized Fourier integral. Important refinements of Povzner's results were given by Kato /3/ and Ikebe /4/.

The stationary scattering problem has not been completely formulated for systems of three or more bodies. This is due to the basic difference between such systems and the two-body system. In the latter both particles move freely before and after collision, which can only alter their courses. Conservation of energy precludes the formation of a bound state. This restriction disappears in the case of three bodies, since the energy released upon formation of a bound pair may be imparted to the third body. This obviously refers also to systems of four, five and more bodies, out of which two or more may become associated in a bound state.

The stationary scattering theory stipulates the existence of a solution of the Schrödinger equation for each of the above possibilities. Therefore, where in the two-body case we have "as many" solutions as there are plane waves, i.e., eigenfunctions of the unperturbed energy operator, in the case of three or more bodies there should be, generally speaking, "more" solutions then plane waves, or eigenfunctions of any operator obtained from the energy operator by deleting some of the interaction terms. In other words, in the two-body case the perturbation does not change the continuous character of the energy spectrum, while in the case of three or more bodies it adds new branches to the continuous spectrum of the unperturbed operator. It is obviously a very tedious and difficult problem to write down all the possible asymptotic forms of solutions of equation (0.2) for $N > 2$.

This problem is fortunately not really crucial, since the described procedure for assembling a complete set of eigenfunctions of the continuous spectrum is not unique. The set of solutions $\psi(x, k)$ in the case $N = 2$, for example, may be obtained as follows. Consider the resolvent of the operator $\mathbf{h}$,

$$\mathbf{r}(z) = (\mathbf{h} - z\mathbf{e})^{-1},$$

where $\mathbf{e}$ is the unit operator, and z a complex number with $\operatorname{Im} z \neq 0$. A. Ya. Povzner [loc. cit.] has shown that $\mathbf{r}(z)$ is an integral operator with a kernel $r(x, y; z)$ which is continuous in x and y for $x \neq y$, and absolutely integrable with respect to y for $\operatorname{Im} z \neq 0$. It also follows from Povzner's results that

$$\psi(x, k) = \lim_{\varepsilon \to +0} -i\varepsilon \int r\left(x, y, \frac{k^2}{2m} + i\varepsilon\right) \exp\{i(k, y)\}\, dy.$$

Thus, we may obtain the eigenfunctions by a suitable limiting transition from the complex plane to the real axis in the kernel of the resolvent of the energy operator. This procedure is closely connected with the time-dependent formulation of the scattering problem, in which it is required to find for the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(x_1, \ldots, x_N, t) = \mathbf{H}_N\psi(x_1, \ldots, x_N, t),$$

solutions satisfying a certain asymptotic condition for $t \to -\infty$.

We shall not go here into the details of the time-dependent scattering problem, but only remark that it is in principle not difficult to formulate it for systems consisting of any number of particles. This furnishes a natural procedure for the derivation of a complete set of eigenfunctions of the continuous spectrum for the given energy operator. In order to carry out and justify this procedure we clearly must first make a detailed study of the behavior of the kernel of the energy operator's resolvent in the neighborhood of the real axis.

This tract is devoted to the investigation of the energy operator for a system of three pairwise-interacting particles, i. e., the simplest system in which the perturbation caused by the interaction alters the character of the continuous spectrum. The main part of the book deals with the investigation of the behavior of the resolvent of the energy operator. The obtained results are used to prove the theorem on expansion in eigenfunctions of this operator, to establish the time-dependent formulation of the scattering problem for our three-body system, and to construct the corresponding unitary scattering operator. We start with an investigation of the energy operator for a two-body system, even though it is quite well known, because this simpler case serves as an illustration of almost all our methods, and because we shall require for the study of the three-body energy operator certain properties of the two-body operator which have not yet been investigated.

Unlike the authors cited above, we shall investigate the energy operators in the momentum representation; the unperturbed operators then become operators of multiplication by a function, and the perturbation is represented by integral operators. The momentum representation treatment of the two-body case leads to somewhat greater analytical difficulties than the configuration representation. These difficulties are concerned with the appearance of singular integrals, the operations with which demand some practice. The author believes, however, that the momentum representation considerably simplifies the study of the energy operator for a three-body system, as the following argument shows. The singularities of the resolvent which appear in the configuration representation when approaching the real axis are due to the too-slow decrease of the kernel at infinity. In the three-body case the kernel of the resolvent may be expected to display a very complicated asymptotic behavior; in particular, it may oscillate in certain directions in configuration space and fall off in others. In contrast, the singularities of the resolvent in momentum representation are poles, and hence easily manageable.

The book consists of eleven sections and four appendices. In § 1 the operators in question are rigorously defined in Hilbert space, i. e., their domain of definition is specified and their self-adjointness proved. Here are also formulated the basic conditions imposed on the potentials which in the rest of the book are tacitly assumed to be fulfilled. §§ 2 and 4, and §§ 3, 5, 6, 7, respectively, deal with the resolvents of energy operators for two and three bodies; the resolvents are expressed in terms of integral operators, and integral equations are set up and investigated in order to derive estimates for the kernels and examine their singularities in the variable z near the real axis. The obtained results are applied in §§ 8 and 9 to the proof of eigenfunction expansion theorems, and in §§ 10 and 11 to the time-dependent formulation of the scattering problem and the construction of the scattering operator. Appendix I gives the derivation of some properties of functions which satisfy the Hölder condition, and of singular integrals containing these functions. Appendices II and III give proofs of estimates of some integrals applied in the text. Appendix IV contains remarks and references to the literature which are not mentioned in the main text.

Throughout, the term "variable" and the letters x, k, p, q with or without indices denote vectors in three-dimensional space. In order to distinguish these from other variables, they are occasionally called three-dimensional variables. The symbol (k, p) designates the scalar product of the vectors k and p; $k^2=(k, k)$, $|k|=(k^2)^{1/2}$; dk and $d\Omega_k$ denote the volume element, and the surface element of the unit sphere, specified by the vectors k and $\frac{k}{|k|}$. The symbol $\int$ is not an indefinite integral but indicates integration over the entire domain of the integration variables. The letter x denotes vectors in configuration space, and k, p, q in momentum space.

The transition from configuration to momentum representation is effected by means of the Fourier transformation

$$f(k)=\int \hat{f}(x) \exp\{i(k, x)\}\, dx.$$

When an estimate is uniform in all the arguments of the functions involved, the constants which appear in it are denoted by C.

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§ 1. The operators $\mathbf{h}$ and $\mathbf{H}$

In this section the energy operators in momentum representation for two- and three-body systems are introduced, formal expressions derived for them, their domains of definition given in the corresponding Hilbert space, and their self-adjointness proved.

We start with the energy operators in configuration representation, defined as differential operators acting on functions of the appropriate number of variables

$$\mathbf{H}_2\psi(x_1, x_2)=\left\{-\frac{1}{2m_1}\nabla_1^2-\frac{1}{2m_2}\nabla_2^2+\hat{v}_{12}(x_1-x_2)\right\}\psi(x_1, x_2);$$

$$\mathbf{H}_3\psi(x_1, x_2, x_3)=\left\{-\frac{1}{2m_1}\nabla_1^2-\frac{1}{2m_2}\nabla_2^2-\frac{1}{2m_3}\nabla_3^2+\hat{v}_{23}(x_2-x_3)+\right.$$
$$\left.+\hat{v}_{31}(x_3-x_1)+\hat{v}_{12}(x_1-x_2)\right\}\psi(x_1, x_2, x_3).$$

Here m_1, m_2, m_3 are the particle masses, ∇_i^2 is the Laplace operator for the variable x_i, and the functions $\hat{v}_{23}(x)$, $\hat{v}_{31}(x)$, $\hat{v}_{12}(x)$represent the two-body interactions.

The passage to momentum representation is accomplished by applying a Fourier transformation. The differentiation operators are thereby transformed into operators of multiplication by the corresponding variable, and the operators of multiplication by a function — into integral operators, so that $\mathbf{H}_2$ and $\mathbf{H}_3$ assume the form

$$\mathbf{H}_2 f(k_1, k_2) = \left\{\frac{1}{2m_1} k_1^2 + \frac{1}{2m_2} k_2^2\right\} f(k_1, k_2) + \mathbf{V}_{12} f(k_1, k_2);$$

$$\mathbf{H}_3 f(k_1, k_2, k_3) = \left\{\frac{1}{2m_1} k_1^2 + \frac{1}{2m_2} k_2^2 + \frac{1}{2m_3} k_3^2\right\} f(k_1, k_2, k_3) + \\ + (\mathbf{V}_{23} + \mathbf{V}_{31} + \mathbf{V}_{12}) f(k_1, k_2, k_3),$$

where $\mathbf{V}_{23}$, $\mathbf{V}_{31}$, $\mathbf{V}_{12}$ are integral operators acting each only on the variables k_2 and k_3, k_3 and k_1, and k_1 and k_2, respectively; thus

$$\mathbf{V}_{12} f(k_1, k_2,) = \int v_{12}\left(\frac{k_1 - k_2 - k_1' + k_2'}{2}\right) \delta(k_1 + k_2 - k_1' - k_2') f(k_1', k_2') dk_1' dk_2'.$$

The operators $\mathbf{H}_2$ and $\mathbf{H}_3$ written in this way contain an inessential term describing the kinetic energy of the center of mass. It is convenient to remove this term, i. e., to investigate the energy operator in the center-of-mass system.

This is done in the two-body case by passing from the variables k_1, k_2 to the new variables

$$K = k_1 + k_2; \quad k = \frac{m_2 k_1 - m_1 k_2}{m_1 + m_2}.$$

The operator $\mathbf{H}_2$ becomes

$$\mathbf{H}_2 f(K, k) = \frac{1}{2M} K^2 f(K, k) + \mathbf{h} f(K, k),$$

where $\mathbf{h}$ operates only on the variable k:

$$\mathbf{h} f(k) = \frac{1}{2m} k^2 f(k) + \int v(k - k') f(k') dk' = (\mathbf{h}_0 + \mathbf{v}) f(k). \tag{1.1}$$

Here

$$M = m_1 + m_2; \quad m = \frac{m_1 m_2}{m_1 + m_2}$$

and $v(k)$ is the Fourier transform of $\hat{v}_{12}(x)$.

In the three-body case this transformation may first be applied to the coordinates of any two of them, and then to their center of mass and the third particle. This leads to the three coordinate systems

$$K = k_1 + k_2 + k_3;$$

$$k_{23} = \frac{m_3 k_2 - m_2 k_3}{m_2 + m_3}; \quad p_1 = \frac{m_1 (k_2 + k_3) - (m_2 + m_3) k_1}{m_1 + m_2 + m_3};$$

$$k_{31} = \frac{m_1 k_3 - m_3 k_1}{m_3 + m_1}; \quad p_2 = \frac{m_2 (k_3 + k_1) - (m_3 + m_1) k_2}{m_1 + m_2 + m_3};$$

$$k_{12} = \frac{m_2 k_1 - m_1 k_2}{m_1 + m_2}; \quad p_3 = \frac{m_3 (k_1 + k_2) - (m_1 + m_2) k_3}{m_1 + m_2 + m_3}.$$

Each of the pairs of variables k_{23}, p_1; k_{31}, p_2; k_{12}, p_3 may be expressed by means of any other pair, so that K, k and p may be chosen as the independent variables. Henceforward we shall simply write k and p, whenever the choice of a particular pair is immaterial.

In these coordinates the operator $\mathbf{H}_3$ splits into two terms

$$\mathbf{H}_3 f(K, k, p) = \frac{1}{2M} K^2 f(K, k, p) + \mathbf{H} f(K, k, p),$$

$$(M = m_1 + m_2 + m_3),$$

where $\mathbf{H}$ operates only on k and p and has the form

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{V}_{23} + \mathbf{V}_{31} + \mathbf{V}_{12}. \tag{1.2}$$

Here $\mathbf{H}_0$ is the operator of multiplication by the function

$$H_0(k, p)=\frac{1}{2m_{23}}k_{23}^2+\frac{1}{2n_1}p_1^2=\frac{1}{2m_{31}}k_{31}^2+\frac{1}{2n_2}p_2^2=\frac{1}{2m_{12}}k_{12}^2+\frac{1}{2n_3}p_3^2,$$

where

$$m_{23}=\frac{m_2m_3}{m_2+m_3};\quad n_1=\frac{m_1(m_2+m_3)}{m_1+m_2+m_3};$$

$$m_{31}=\frac{m_3m_1}{m_3+m_1};\quad n_2=\frac{m_2(m_3+m_1)}{m_1+m_2+m_3};$$

$$m_{12}=\frac{m_1m_2}{m_1+m_2};\quad n_3=\frac{m_3(m_1+m_2)}{m_1+m_2+m_3};$$

the operator $\mathbf{V}_{23}$ is an integral operator with the kernel

$$\mathscr{V}_{23}(k, p; k', p')=v_{23}(k_{23}-k'_{23})\delta(p_1-p'_1),$$

i. e. ,

$$\mathbf{V}_{23}f(k, p)=\int v_{23}(k_{23}-k'_{23})f(k'_{23}, p_1)dk'_{23};$$

the operators $\mathbf{V}_{31}$ and $\mathbf{V}_{12}$ are similarly defined in the coordinates k_{31}, p_2 and k_{12}, p_3, respectively.

The function $H_0(k, p)$ will frequently be written in the form

$$H_0(k, p)=\frac{1}{2m}k^2+\frac{1}{2n}p^2,$$

without any indices.

It is sometimes convenient to use as independent variables instead of k, p any two of the variables p_1, p_2, p_3, e. g. , p_2, p_1 instead of k_{23}, p_1. The relations between the k-type and the p-type variables are

$$k_{23}=-p_2-\frac{m_2}{m_2+m_3}p_1=p_3+\frac{m_3}{m_2+m_3}p_1;$$

$$k_{31}=-p_3-\frac{m_3}{m_1+m_3}p_2=p_1+\frac{m_1}{m_1+m_3}p_2;$$

$$k_{12}=-p_1-\frac{m_1}{m_1+m_2}p_3=p_2+\frac{m_2}{m_1+m_2}p_3;$$

To pass from any pair of p_1, p_2, p_3 to any other pair we make use of the relation

$$p_1+p_2+p_3=0.$$

It is not difficult to express the kernels of the operators $\mathbf{V}_{23}$, $\mathbf{V}_{31}$, $\mathbf{V}_{12}$ in terms of these variables. Thus, for example, if $p_1=p'_1$, then $k_{23}-k'_{23}=$ $=-p_2+p'_2=p_3-p'_3$, so that

$$\mathscr{V}_{23}(k, p; k', p')=v_{23}(-p_2+p'_2)\delta(p_1-p'_1)=v_{23}(p_3-p'_3)\delta(p_1-p'_1).$$

The function $H_0(k, p)$ is conveniently expressed in terms of p_1, p_2, p_3 as

$$H_0(k, p)=\frac{p_1^2}{2m_{31}}+\frac{(p_1, p_2)}{m_3}+\frac{p_2^2}{2m_{23}}=$$

$$=\frac{p_2^2}{2m_{12}}+\frac{(p_2, p_3)}{m_1}+\frac{p_3^2}{2m_{31}}=\frac{p_3^2}{2m_{23}}+\frac{(p_2, p_1)}{m_2}+\frac{p_1^2}{2m_{12}}.$$

We assign to the formal expressions (1.1) and (1.2) self-adjoint operators $\mathbf{h}$ and $\mathbf{H}$ acting in the Hilbert space of square-integrable functions of the suitable number of variables.

We denote by $\mathfrak{h}$ the Hilbert space of the functions $f(k)$ with the usual scalar product

$$(f, f')=\int f(k)\overline{f'(k)}dk.$$

The Hilbert space of the functions of two variables $f(k, p)$ with the scalar product

$$(f, f') = \int f(k, p)\overline{f'(k, p)}\, dk dp$$

will be denoted by $\mathfrak{H}$.

We do not introduce different notations for the components, the scalar product and the norm in $\mathfrak{h}$ and $\mathfrak{H}$, since it will always be evident which space is meant.

The functions $v(k)$, $v_{23}(k)$, $v_{31}(k)$, $v_{12}(k)$ will be assumed to satisfy the following conditions (here exemplified in the case of $v(k)$): boundedness and sufficiently fast falling off

$$|v(k)| \leqslant C(1+|k|)^{-1-\theta_0}, \qquad (\theta_0 > 0), \qquad (A_{\theta_0})$$

smoothness

$$|v(k) - v(k+h)| \leqslant C(1+|k|)^{-1-\theta_0}|h|^{\mu_0}; \; (|h| \leqslant 1, \; \mu_0 > 0) \qquad (B_{\mu_0})$$

and real-valuedness

$$v(-k) = \overline{v(k)}. \qquad (R)$$

The set of functions $f(k)$, dense in $\mathfrak{h}$, which satisfy the condition

$$\int (1+k^2)^2 |f(k)|^2 dk < \infty,$$

is denoted by $\mathfrak{d}$. The set of functions $f(k, p)$, dense in $\mathfrak{H}$, which satisfy the condition

$$\int (1+k^2+p^2)^2 |f(k, p)|^2 dk dp < \infty.$$

is denoted by $\mathfrak{D}$.

Theorem 1.1. *Let the function* $v(k)$ *satisfy conditions* A_{θ_0} *and* R, *with* $\theta_0 > \frac{1}{2}$. *Then the expression (1.1), defined on* $\mathfrak{d}$, *defines a self-adjoint operator* $\mathbf{h}$ *in* $\mathfrak{h}$.

Let the functions $v_{23}(k)$, $v_{31}(k)$, $v_{12}(k)$ *satisfy conditions* A_{θ_0} *and* R, *with* $\theta_0 > \frac{1}{2}$. *Then the expression (1.2), defined on* $\mathfrak{D}$, *together with its subsequent relations, defines a self-adjoint operator* $\mathbf{H}$ *in* $\mathfrak{H}$.

We remark that this theorem is a particular case of a well-known theorem of Kato /3/ on the self-adjointness of the many-body Schrödinger operator. Kato's conditions are formulated in the configuration representation. These conditions are known to be fulfilled if all the potentials $\tilde{v}_{ij}(x)$ are square-integrable in the entire three-dimensional space. If the exponent θ_0 in condition A_{θ_0} is larger than $\frac{1}{2}$, then the functions $v(k)$, and consequently also their Fourier transforms, are square-integrable over the entire space and therefore satisfy Kato's criterion. Nonetheless, we shall go ahead and prove Theorem 1.1 for the sake of completeness of presentation. The proof is based on the following general proposition

Lemma 1.1. *Let a self-adjoint operator* $\mathbf{A}$, *with the domain of definition* $\mathfrak{D}(\mathbf{A})$, *be given in the Hilbert space* $\mathfrak{H}$. *Let a symmetric operator* $\mathbf{B}$, *defined on* $\mathfrak{D}(\mathbf{A})$, *satisfy the condition*

$$\|\mathbf{B}f\| \leqslant \delta\|\mathbf{A}f\| + C\|f\|,$$

where $\delta < 1$. *Then the operator*

$$\tilde{\mathbf{A}} = \mathbf{A} + \mathbf{B}$$

with the domain of definition $\mathfrak{D}(\tilde{\mathbf{A}}) = \mathfrak{D}(\mathbf{A})$ *is self-adjoint.*

This lemma plays a central part in the proof of Kato's theorem and is proved in /3/.

Let us apply this lemma to the proof of Theorem 1.1, taking the operators $\mathbf{h}_0$ or $\mathbf{H}_0$ as $\mathbf{A}$, and $\mathbf{v}$ or $\mathbf{V}=\mathbf{V}_{23}+\mathbf{V}_{31}+\mathbf{V}_{12}$ as $\mathbf{B}$, so that $\tilde{\mathbf{A}}$ corresponds to $\mathbf{h}$ or $\mathbf{H}$, respectively. The domains $\mathfrak{d}$ and $\mathfrak{D}$ were chosen above as the natural domains of definition of the self-adjoint operators of multiplication by a function $\mathbf{h}_0$ and $\mathbf{H}_0$.

Lemma 1.2. Let the conditions A_{θ_0} *and* R, *with* $\theta_0 > \frac{1}{2}$, *be fulfilled. Then the operator* $\mathbf{v}$ *is defined and symmetric on* $\mathfrak{d}$, *and*

$$\|\mathbf{v}f\| \leqslant \delta \|\mathbf{h}_0 f\| + C(\delta)\|f\|, \qquad (1.3)$$

where δ *may be made as small as desired.*

Let us first prove that there exists a δ, such that the operator $\mathbf{v}$ is subordinate to the operator $(\mathbf{e}+\mathbf{h}_0)^{1-\delta}$. Let $f \in \mathfrak{d}$, then

$$|\mathbf{v}f(k)|^2 = \left| \int v(k-k')f(k')\,dk' \right|^2 \leqslant$$

$$\leqslant C\left(\int \frac{1}{(1+|k-k'|)^{1+\theta_0}} \frac{(1+k'^2)^{1-\delta}}{(1+k'^2)^{1-\delta}} \cdot |f(k')|\,dk' \right)^2 \leqslant$$

$$\leqslant C\int (1+k'^2)^{-2+2\delta}(1+|k-k'|)^{-2(1+\theta_0)}\,dk' \|(\mathbf{e}+\mathbf{h}_0)^{1-\delta} f\|^2.$$

If $\theta_0 > \frac{1}{2}$, then $2(1+\theta_0)>3$, and we deduce that if δ is such that $2(2-\delta)>3$, i. e., $\delta<\frac{1}{4}$, then

$$\|\mathbf{v}f\| \leqslant C\|(\mathbf{e}+\mathbf{h}_0)^{1-\delta} f\|. \qquad (1.4)$$

Inequality (1.3) follows from (1.4) in view of the elementary inequality

$$(1+x)^{1-\delta} \leqslant \delta x + \left(\frac{1}{\delta}\right)^{1/\delta} (x>0;\ 0<\delta<1).$$

In order to prove the symmetry of $\mathbf{v}$ it is sufficient, on account of condition **R**, to show that the integral

$$I = \int f(k)\,v(k-k')\overline{f'(k')}\,dk\,dk'$$

converges absolutely for any f and f' in $\mathfrak{d}$. We note that the functions $f(k)$ in $\mathfrak{d}$ are absolutely integrable. In fact

$$\left(\int |f(k)|\,dk\right)^2 \leqslant \int (1+k^2)^2 |f(k)|^2\,dk \int \frac{dk}{(1+k^2)^2} < \infty.$$

This immediately implies the absolute convergence of I. The lemma is thereby proved.

We note that the operator $\mathbf{H}_0$ may be expressed as a sum of two nonnegative terms, e. g.,

$$\mathbf{H}_0 = \mathbf{h}_0^{(23)} + \mathbf{h}_0^{(1)},$$

where

$$\mathbf{h}_0^{(23)} f(k,\ p) = \frac{1}{2m_{23}} k_{23}^2 f(k,\ p);\ \mathbf{h}_0^{(1)} f(k,\ p) = \frac{1}{2n_1} p_1^2 f(k,\ p),$$

Repeating the proof of Lemma 1.2, one may show that

$$\|\mathbf{V}_{23} f\| \leqslant \delta \|\mathbf{h}_0^{(23)} f\| + C(\delta)\|f\| \leqslant \delta \|\mathbf{H}_0 f\| + C(\delta)\|f\|$$

for any $f \in \mathfrak{D}$ and that $\mathbf{V}_{23}$ is symmetric in $\mathfrak{D}$. The operators $\mathbf{V}_{31}$ and $\mathbf{V}_{12}$ are treated in the same way. Theorem 1.1 follows from Lemmas 1.1, 1.2 and the preceding arguments.

§ 2. Construction of the resolvent of the operator $\mathbf{h}$ for $\operatorname{Im} z \neq 0$

In this section we begin the investigation of the resolvent of the operator $\mathbf{h}$, i. e., of the operator

$$\mathbf{r}(z) = (\mathbf{h} - z\mathbf{e})^{-1}, \tag{2.1}$$

where $\mathbf{e}$ is the unit operator in $\mathfrak{h}$, and z a complex number. The resolvent $\mathbf{r}(z)$ is defined and bounded for any z with $\operatorname{Im} z \neq 0$, on account of the self-adjointness of $\mathbf{h}$. Its range is the same for all such z and coincides with $\mathfrak{d}$.

The operator $\mathbf{r}(z)$ is mainly studied by means of the equation relating it to the operators $\mathbf{v}$ and the resolvent $\mathbf{r}_0(z)$ of $\mathbf{h}_0$,

$$\mathbf{r}_0(z) = (\mathbf{h}_0 - z\mathbf{e})^{-1}. \tag{2.2}$$

This well-known equation has the form

$$\mathbf{r}(z) = \mathbf{r}_0(z) - \mathbf{r}_0(z)\mathbf{v}\mathbf{r}(z) \tag{2.3}$$

and determines $\mathbf{r}(z)$ uniquely. This is more precisely expressed as follows:

Lemma 2.1. Let the bounded operator $\tilde{\mathbf{r}}$, *whose range is contained in* $\mathfrak{d}$, *satisfy equation (2.3) for some* z, $\operatorname{Im} z \neq 0$. *Then* $\tilde{\mathbf{r}} = \mathbf{r}(z)$.

To prove this, consider the operator $\mathbf{r}_1 = \mathbf{r}(z) - \tilde{\mathbf{r}}$. It satisfies the equation

$$\mathbf{r}_1 = -\mathbf{r}_0(z)\mathbf{v}\mathbf{r}_1 \tag{2.4}$$

for the above-defined z. For any $f \in \mathfrak{h}$ there exists a $g = \mathbf{r}_1 f$ with $g \in \mathfrak{d}$. Equation (2.4) yields the following equation for g

$$g = -\mathbf{r}_0(z)\mathbf{v}g. \tag{2.5}$$

Multiplying (2.5) by $\mathbf{h}_0 - z\mathbf{e}$,

$$\mathbf{h}_0 g - zg = -\mathbf{v}g \tag{2.6}$$

or

$$\mathbf{h}g = zg. \tag{2.7}$$

The self-adjointness of $\mathbf{h}$ implies that $g = 0$ and hence $\mathbf{r}_1 = 0$, i. e., $\mathbf{r} = \mathbf{r}(z)$. This proves the lemma.

It is convenient to consider the operator

$$\mathbf{t}(z) = \mathbf{v} - \mathbf{v}\mathbf{r}(z)\mathbf{v}. \tag{2.8}$$

This operator is defined on $\mathfrak{d}$ for all z with $\operatorname{Im} z \neq 0$, and satisfies the equation

$$\mathbf{t}(z) = \mathbf{v} - \mathbf{v}\mathbf{r}_0(z)\mathbf{t}(z). \tag{2.9}$$

The operator $\mathbf{r}(z)$ is expressed through $\mathbf{t}(z)$ as

$$\mathbf{r}(z) = \mathbf{r}_0(z) - \mathbf{r}_0(z)\mathbf{t}(z)\mathbf{r}_0(z). \tag{2.10}$$

Indeed, multiplying (2.3) on the left and on the right by $\mathbf{v}$, we obtain

$$\mathbf{t}(z) - \mathbf{v} = -\mathbf{v}\mathbf{r}_0(z)\mathbf{v} + \mathbf{v}\mathbf{r}_0(z)\mathbf{v}\mathbf{r}(z)\mathbf{v} =$$
$$= -\mathbf{v}\mathbf{r}_0(z)\mathbf{v} + \mathbf{v}\mathbf{r}_0(z)[-\mathbf{t}(z) + \mathbf{v}] = -\mathbf{v}\mathbf{r}_0(z)\mathbf{t}(z).$$

It is similarly verified that the operator $\mathbf{r}(z)$, constructed from $\mathbf{t}(z)$ according to (2.10), satisfies equation (2.3), and that its range is contained within $\mathfrak{d}$. By Lemma 2.1, this operator coincides with the resolvent $\mathbf{r}(z)$.

It follows from Lemma 2.1 and the preceding arguments that

Lemma 2.2. Let the operator $\tilde{\mathbf{t}}$, *defined on* $\mathfrak{D}$, *satisfy equation (2.9) for some* z. *Then* $\tilde{\mathbf{t}}=\mathbf{t}(z)$.

It is more convenient to study the operator $\mathbf{t}(z)$ than the resolvent $\mathbf{r}(z)$, since this operator may be expected in view of (2.8) to be, like $\mathbf{v}$, an integral operator with a smooth and bounded kernel. To prove the last statement rigorously, consider the integral equation

$$t(k, k', z)=v(k-k')-\int v(k-k'')\left(\frac{k''^2}{2m}-z\right)^{-1} t(k'', k', z)\,dk'' \qquad (2.11)$$

for the kernel $t(k, k', z)$. This equation is studied in detail in § 4 under the assumption that conditions A_{θ_0}, B_{μ_0} and R are fulfilled. In particular, it is proved there that:

Equation (2.11) has for any z *with* $\operatorname{Im} z\neq 0$ *a solution* $t(k, k', z)$, *continuous in* k, k' *and* z, *and estimated by*

$$|t(k, k', z)|\leqslant C(1+|k-k'|)^{-1-\theta}, \quad \theta<\theta_0 \qquad (2.12)$$

uniformly in z *with* $|\operatorname{Im} z|>\delta$, $\delta>0$, *and any* k *and* k'.

We construct from the kernel $t(k, k', z)$ the operator $\tilde{\mathbf{t}}(z)$:

$$\tilde{\mathbf{t}}(z)f(k)=\int t(k, k', z)f(k')\,dk'.$$

Here θ in condition (2.12) may be taken to be less than $\frac{1}{2}$. Repeating the steps of the proof (§ 1) that the operator $\mathbf{v}$ is defined on $\mathfrak{D}$, we may prove the same for $\tilde{\mathbf{t}}(z)$. It follows from (2.11) that $\tilde{\mathbf{t}}(z)$ satisfies (2.9); hence, by Lemma 2.2 it coincides with $\mathbf{t}(z)$.

We can now infer

Theorem 2.1. Let $\operatorname{Im} z\neq 0$. *The resolvent* $\mathbf{r}(z)$ *is represented in the form (2.10), where* $\mathbf{t}(z)$ *is an integral operator, and its kernel* $t(k, k', z)$ *is continuous in all its variables and satisfies the estimate (2.12).*

The characteristic properties of the resolvent of a self-adjoint operator are known to be

$$\mathbf{r}^*(z)=\mathbf{r}(\bar{z}) \qquad (2.13)$$

and

$$\mathbf{r}(z_1)-\mathbf{r}(z_2)=(z_1-z_2)\mathbf{r}(z_1)\mathbf{r}(z_2). \qquad (2.14)$$

The last relation is frequently called Hilbert's identity.

Let us see what properties of the kernel $t(k, k', z)$ are implied by relations (2.13) and (2.14).

Lemma 2.3. The following relation is valid

$$t(k, k', z)=\overline{t(k', k, \bar{z})}. \qquad (2.15)$$

Proof. Let f and f' be some functions in $\mathfrak{D}$. Since $\mathbf{v}$ is symmetric on $\mathfrak{D}$, we have by (2.8) and (2.13)

$$(\mathbf{t}(z)f, f')=(f, \mathbf{t}(\bar{z})f'),$$

that is,

$$\int\left\{\int t(k, k', z)f(k')\,dk'\right\}\overline{f'(k)}\,dk=\int\overline{\left\{\int t(k', k, \bar{z})f'(k)\,dk\right\}}f(k')\,dk'. \qquad (2.16)$$

Since these integrals converge absolutely, we may change the order of integration on the right-hand side. (2.15) follows from (2.16) since $f(k)$ and $f'(k)$ are arbitrary. This proves the lemma.

Lemma 2.4. The following relation is valid

$$t(k, k', z_1) - t(k, k', z_2) =$$
$$= (z_2 - z_1) \int t(k, k'', z_1) \frac{1}{\frac{k''^2}{2m} - z_1} \frac{1}{\frac{k''^2}{2m} - z_2} t(k'', k', z_2) dk''. \tag{2.17}$$

Proof. We first prove that the following relations hold

$$\mathbf{t}(z)\mathbf{r}_0(z) = \mathbf{v}\mathbf{r}(z); \quad \mathbf{r}_0(z)\mathbf{t}(z) = \mathbf{r}(z)\mathbf{v}. \tag{2.18}$$

Multiplying (2.10) on the left by $\mathbf{v}$, we obtain

$$\mathbf{v}\mathbf{r}(z) = \mathbf{v}\mathbf{r}_0(z) - \mathbf{v}\mathbf{r}_0(z)\mathbf{t}(z)\mathbf{r}_0(z) = [\mathbf{v} - \mathbf{v}\mathbf{r}_0(z)\mathbf{t}(z)]\mathbf{r}_0(z) = \mathbf{t}(z)\mathbf{r}_0(z).$$

The second relation in (2.18) follows on multiplying (2.8) on the left by $\mathbf{r}_0(z)$. We now multiply (2.14) on the left and right by $\mathbf{v}$, obtaining

$$\mathbf{t}(z_1) - \mathbf{t}(z_2) = -(z_1 - z_2)\mathbf{v}\mathbf{r}(z_1)\mathbf{r}(z_2)\mathbf{v} = (z_2 - z_1)\mathbf{t}(z_1)\mathbf{r}_0(z_1)\mathbf{r}_0(z_2)\mathbf{t}(z_2).$$

This implies (2.17). The rigorous proof duplicates that of Lemma 2.3.

§ 3. Construction of the resolvent of the operator $\mathbf{H}$ for complex z

In this section we begin the investigation of the resolvent of the operator $\mathbf{H}$. We introduce the notation

$$\mathbf{R}(z) = (\mathbf{H} - z\mathbf{E})^{-1}; \quad \mathbf{R}_0(z) = (\mathbf{H}_0 - z\mathbf{E})^{-1}. \tag{3.1}$$

(Here $\mathbf{E}$ is the unit operator in $\mathfrak{H}$.) These operators are defined for all z with $\operatorname{Im} z \neq 0$ and map $\mathfrak{H}$ onto $\mathfrak{D}$. The following equation holds

$$\mathbf{R}(z) = \mathbf{R}_0(z) - \mathbf{R}_0(z)\mathbf{V}\mathbf{R}(z), \tag{3.2}$$

where

$$\mathbf{V} = \mathbf{V}_{23} + \mathbf{V}_{31} + \mathbf{V}_{12}. \tag{3.3}$$

Lemma 2.1 may now be reformulated as follows:

Lemma 3.1. Let the bounded operator $\tilde{\mathbf{R}}$, *with a range contained in* $\mathfrak{D}$, *satisfy equation (3.2) for some* z. *Then* $\tilde{\mathbf{R}} = \mathbf{R}(z)$.

We denote by $\mathbf{T}(z)$ the operator defined on $\mathfrak{D}$ by the relation

$$\mathbf{T}(z) = \mathbf{V} - \mathbf{V}\mathbf{R}(z)\mathbf{V}. \tag{3.4}$$

The resolvent $\mathbf{R}(z)$ is expressed through $\mathbf{T}(z)$ as

$$\mathbf{R}(z) = \mathbf{R}_0(z) - \mathbf{R}_0(z)\mathbf{T}(z)\mathbf{R}_0(z). \tag{3.5}$$

The operator $\mathbf{T}(z)$ satisfies the equation

$$\mathbf{T}(z) = \mathbf{V} - \mathbf{V}\mathbf{R}_0(z)\mathbf{T}(z). \tag{3.6}$$

Any operator $\tilde{\mathbf{T}}$ with a range lying in $\mathfrak{D}$ which satisfies equation (3.6) for some z, coincides with $\mathbf{T}(z)$. All these propositions are proved exactly as their counterparts in § 2. Equation (3.6), however, is not very useful for the investigation of the operator $\mathbf{T}(z)$. The reason for this is that the operator $\mathbf{V}_{23}\mathbf{R}_0(z)$ which appears in this equation has the kernel

$$v_{23}(k_{23} - k'_{23})\delta(p_1 - p'_1)\left(\frac{k^2}{2m} + \frac{p^2}{2n} - z\right)^{-1},$$

containing a δ-function, which is not removed in iterations of (3.6). In fact, the kernel of $\mathbf{V}_{23}\mathbf{R}_0(z)\mathbf{V}_{23}\mathbf{R}_0(z)$ contains the same δ-function as the kernel of $\mathbf{V}_{23}\mathbf{R}_0(z)$. It is therefore impossible to reduce equation (3.6) to an equation with a completely continuous operator in any function space.

We note, however, that these δ-functions disappear in the product of the operators $\mathbf{V}_{23}\mathbf{R}_0(z)$ and $\mathbf{V}_{31}\mathbf{R}_0(z)$ upon iterating equation (3.6). Actually, one may take p_1 and p_2 as integration variables in the kernel of $\mathbf{V}_{23}\mathbf{R}_0(z)\mathbf{V}_{31}\mathbf{R}_0(z)$, and the δ-functions are removed by integration. The same occurs in the products of any two operators of the type $\mathbf{V}_{23}\mathbf{R}_0(z)$ with different indices. These arguments point the way to the construction of suitable equations for the investigation of the resolvent.

Consider the operators

$$\mathbf{M}_{\alpha\beta}(z)=\delta_{\alpha\beta}\mathbf{V}_\alpha-\mathbf{V}_\alpha\mathbf{R}(z)\mathbf{V}_\beta. \tag{3.7}$$

We have introduced for brevity the indices α, β which run through the values 23, 31, 12; $\delta_{\alpha\beta}$ is the Kronecker symbol. It is easily verified that our operator $\mathbf{T}(z)$ is expressed by the $\mathbf{M}_{\alpha\beta}(z)$ as

$$\mathbf{T}(z)=\sum_{\alpha,\beta}\mathbf{M}_{\alpha\beta}(z). \tag{3.8}$$

Equation (3.2) may be used to show that the operators $\mathbf{M}_{\alpha\beta}(z)$ satisfy the equations

$$\mathbf{M}_{\alpha\beta}(z)=\delta_{\alpha\beta}\mathbf{V}_\alpha-\mathbf{V}_\alpha\mathbf{R}_0(z)\sum_{\gamma}\mathbf{M}_{\gamma\beta}(z). \tag{3.9}$$

The system of equations (3.9) is in no way better then equation (3.6). However, we may perform the following transformation. We split the sum on the right-hand side into two parts

$$\mathbf{V}_\alpha\mathbf{R}_0(z)\mathbf{M}_{\alpha\beta}(z)+\mathbf{V}_\alpha\mathbf{R}_0(z)\sum_{\gamma\neq\alpha}\mathbf{M}_{\gamma\beta}(z)$$

and transfer the first to the left-hand side of (3.9), obtaining

$$[\mathbf{E}+\mathbf{V}_\alpha\mathbf{R}_0(z)]\mathbf{M}_{\alpha\beta}(z)=\delta_{\alpha\beta}\mathbf{V}_\alpha-\mathbf{V}_\alpha\mathbf{R}_0(z)\sum_{\gamma\neq\alpha}\mathbf{M}_{\gamma\beta}(z). \tag{3.10}$$

Consider the operator $\mathbf{T}_\alpha(z)$, defined by the relation

$$\mathbf{T}_\alpha(z)f(k,\ p)=\int t_\alpha\left(k_\alpha,\ k'_\alpha,\ z-\frac{p_\alpha^2}{2n_\alpha}\right)f(k'_\alpha,\ p_\alpha)\,dk'_\alpha. \tag{3.11}$$

Here $t_\alpha(k,\ k',\ z)$ is the solution of equation (2.11) with m_α and $v_\alpha(k)$ for m and $v(k)$. The index α which refers to p and n runs through the values 1, 2, 3 instead of 23, 31, 12, respectively. The estimate (2.12) shows that the operator $\mathbf{T}_\alpha(z)$ is defined on $\mathfrak{D}$ (exactly as we did in § 1 for $\mathbf{V}_\alpha$). It readily follows from (2.11) that $\mathbf{T}_\alpha(z)$ satisfies the equation

$$\mathbf{T}_\alpha(z)=\mathbf{V}_\alpha-\mathbf{V}_\alpha\mathbf{R}_0(z)\mathbf{T}_\alpha(z)=\mathbf{V}_\alpha-\mathbf{T}_\alpha(z)\mathbf{R}_0(z)\mathbf{V}_\alpha. \tag{3.12}$$

We multiply (3.10) by $\mathbf{E}-\mathbf{T}_\alpha(z)\mathbf{R}_0(z)$. By (3.12), $\mathbf{M}_{\alpha\beta}(z)$ must satisfy the system of equations

$$\mathbf{M}_{\alpha\beta}(z)=\delta_{\alpha\beta}\mathbf{T}_\alpha(z)-\mathbf{T}_\alpha(z)\mathbf{R}_0(z)\sum_{\gamma\neq\alpha}\mathbf{M}_{\gamma\beta}(z), \tag{3.13}$$

which is our basic tool in the investigation of the resolvent $\mathbf{R}(z)$.

Lemma 3.2. Let $\mathbf{M}_{\alpha\beta}(z)$ *be nine operators, defined on* $\mathfrak{D}$ *and satisfying the system (3.13). Then the operator*

$$\tilde{\mathbf{T}}(z)=\sum_{\alpha,\beta}\mathbf{M}_{\alpha\beta}(z)$$

coincides with $\mathbf{T}(z)$.

Proof. We apply the arguments which led to the system (3.13). Multiplying (3.13) by the bounded operator $\mathbf{E}+\mathbf{V}_\alpha\mathbf{R}_0(z)$, and bearing (3.12) in mind, we deduce that the operators $\mathbf{M}_{\alpha\beta}(z)$ satisfy (3.9), and hence that $\tilde{\mathbf{T}}(z)$ satisfies (3.6). The required result follows on account of the uniqueness of solution of this equation within the class of operators defined on $\mathfrak{D}$. The lemma is thereby proved.

The kernels of the free terms in (3.13) contain δ-functions. We therefore consider instead of the operators $\mathbf{M}_{\alpha\beta}(z)$ the operators

$$\mathbf{W}_{\alpha\beta}(z)=\mathbf{M}_{\alpha\beta}(z)-\delta_{\alpha\beta}\mathbf{T}_\alpha(z), \tag{3.14}$$

which satisfy a system resembling (3.13),

$$\mathbf{W}_{\alpha\beta}(z)=\mathbf{W}^{(0)}_{\alpha\beta}(z)-\mathbf{T}_\alpha(z)\mathbf{R}_0(z)\sum_{\gamma\neq\alpha}\mathbf{W}_{\gamma\beta}(z), \tag{3.15}$$

but with a different free term

$$\mathbf{W}^{(0)}_{\alpha\alpha}(z)=0;\quad \mathbf{W}^{(0)}_{\alpha\beta}(z)=-\mathbf{T}_\alpha(z)\mathbf{R}_0(z)\mathbf{T}_\beta(z)=\mathbf{Q}^{(1)}_{\alpha\beta}(z). \tag{3.16}$$

We shall use the following notation for the iterations of (3.15)

$$\mathbf{Q}^{(n)}_{\gamma_1,\ldots,\gamma_{n+1}}(z)=(-1)^n\mathbf{T}_{\gamma_1}(z)\mathbf{R}_0(z)\mathbf{T}_{\gamma_2}(z)\ldots\mathbf{T}_{\gamma_n}(z)\mathbf{R}_0(z)\mathbf{T}_{\gamma_{n+1}}(z), \tag{3.17}$$

where the indices $\gamma_1, \ldots, \gamma_{n+1}$ take on values complying with the condition

$$\gamma_i\neq\gamma_{i+1},\quad i=1, 2, \ldots, n. \tag{3.18}$$

It is easily verified that the kernels of the free terms in (3.15) are smooth bounded functions. Thus, e. g.,

$$\begin{aligned}\mathcal{Q}^{(1)}_{23,31}(k, p; k', p'; z)&=t_{23}\left(k_{23}, -p_2'-\frac{m_2}{m_2+m_3}p_1;\ z-\frac{p_1^2}{2n_1}\right)\times\\ &\times\left(\frac{p_1^2}{2m_{31}}+\frac{(p_1, p_2')}{m_3}+\frac{p_2'^2}{2m_{23}}-z\right)^{-1}\times\\ &\times t_{31}\left(p_1+\frac{m_1}{m_1+m_3}p_2', k_{31}', z-\frac{p_2'^2}{2n_2}\right),\end{aligned} \tag{3.19}$$

where the denominator does not vanish for $\operatorname{Im} z\neq 0$. It may be similarly verified that the kernel of any operator $\mathbf{Q}^{(n)}_{\gamma_1,\ldots,\gamma_{n+1}}(z)$ is also a bounded smooth function. The kernels of the operators $\mathbf{W}_{\alpha\beta}(z)$ can also be expected to possess these properties; that they in fact do possess them is proved as follows. Consider the system of integral equations

$$\begin{aligned}\mathcal{W}_{\alpha\beta}(k, p; k', p'; z)&=\mathcal{W}^{(0)}_{\alpha\beta}(k, p; k', p'; z)-\\ &-\int t_\alpha\left(k_\alpha, k_\alpha'', z-\frac{p_\alpha^2}{2n_\alpha}\right)\frac{\delta(p_\alpha-p_\alpha'')}{\frac{k''^2}{2m}+\frac{p''^2}{2n}-z}\sum_{\gamma\neq\alpha}\mathcal{W}_{\gamma\beta}(k'', p''; k', p'; z)\,dk''dp''.\end{aligned} \tag{3.20}$$

This system will be studied in detail below, from § 5 onwards. Let us introduce at this stage some notations in order to formulate the required result which is derived there. We denote by $\tilde{\mathbf{W}}_{\alpha\beta}(z)$ the operators obtained from

$\mathbf{W}_{\alpha\beta}(z)$ upon subtraction of the first three iterations of the system (3.15)

$$\widetilde{\mathbf{W}}_{\alpha\beta}(z)=\mathbf{W}_{\alpha\beta}(z)-\mathbf{Q}^{(1)}_{\alpha\beta}(z)-\mathbf{Q}^{(2)}_{\alpha\beta}(z)-\mathbf{Q}^{(3)}_{\alpha\beta}(z), \tag{3.21}$$

where

$$\mathbf{Q}^{(2)}_{\alpha\beta}(z)=\sum_{\gamma}{}' \mathbf{Q}^{(2)}_{\alpha,\gamma,\beta}(z);\quad \mathbf{Q}^{(3)}_{\alpha\beta}(z)=\sum_{\gamma_1,\gamma_2}{}' \mathbf{Q}^{(3)}_{\alpha,\gamma_1,\gamma_2,\beta}. \tag{3.22}$$

The prime on $\sum$ signifies that summation is carried out over the allowed values of γ, γ_1, γ_2 only, i. e., $\gamma\neq\alpha$, $\gamma\neq\beta$, $\gamma_1\neq\alpha$, $\gamma_1\neq\gamma_2$, $\gamma_2\neq\beta$. We denote by $N(k, p;\ \theta)$ the estimating function

$$N(k, p;\ \theta)=\sum_{\substack{\alpha,\beta\\ \alpha\neq\beta}}(1+|p_\alpha|)^{-1-\theta}(1+|p_\beta|)^{-1-\theta}, \tag{3.23}$$

where p_1, p_2, p_3 are understood to be expressed through any pair of independent variables.

In § 5 we prove the following statement:

The system (3.20) possesses a solution which is continuous in all variables, and such that the kernels $\mathcal{W}_{\alpha\beta}(k, p;\ k', p';\ z)$ *of the operators* $\widetilde{\mathbf{W}}_{\alpha\beta}(z)$ *satisfy the estimates*

$$|\mathcal{W}_{\alpha\beta}(k, p;\ k', p';\ z)|\leqslant CN(k, p;\ \theta)(1+p'^2_\beta)^{-1} \tag{3.24}$$

for any z *belonging to a finite domain that does not border on the real axis, where* θ *may be chosen as close as desired to the* θ_0 *of the condition* A_{θ_0}.

We will now show that the integral operators $\mathbf{W}_{\alpha\beta}(z)$ with the kernels $\mathcal{W}_{\alpha\beta}(k, p;\ k', p';\ z)$ are defined on $\mathfrak{D}$. To this end we make use of the following proposition:

Lemma 3.3. Let $\mathbf{W}$ *be the integral operator*

$$\mathbf{W}f(k, p)=\int\mathcal{W}(k, p;\ k', p')f(k', p')\,dk'dp',$$

whose kernel $\mathcal{W}(k, p;\ k', p')$ *satisfies the estimate*

$$|\mathcal{W}(k, p;\ k', p')|\leqslant CN(k, p;\ \theta)(1+p'^2_\beta)^{-1},$$

where β *may take on any of the values 1, 2, 3, and* $\theta>\frac{1}{2}$. *Then* $\mathbf{W}$ *is defined on* $\mathfrak{D}$ *as an operator in* $\mathfrak{H}$.

Proof. Let $f(k, p)$ be a finite smooth function. Taking k_β and p_β as integration variables, we obtain

$$|\mathbf{W}f(k, p)|^2\leqslant$$
$$\leqslant C(N(k, p;\ \theta))^2\int\frac{dkdp}{(1+p^2)^2(1+k^2+p^2)^2}\int(1+k^2+p^2)^2|f(k, p)|^2\,dkdp\leqslant$$
$$\leqslant C(N(k, p;\ \theta))^2\|(\mathbf{E}+\mathbf{H}_0)f\|^2$$

Since $\theta>\frac{1}{2}$, we get

$$\|\mathbf{W}f\|\leqslant C\|(\mathbf{E}+\mathbf{H}_0)f\|,$$

which shows the operator $\mathbf{W}$ to be defined for any $f\in\mathfrak{D}$. This proves the lemma.

All the operators $\mathbf{Q}^{(n)}_{\gamma_1,\ldots,\gamma_{n+1}}(z)$ are defined on $\mathfrak{D}$. It follows from (3.24) and Lemma 3.3 that the operators $\widetilde{\mathbf{W}}_{\alpha\beta}(z)$, and hence $\mathbf{W}_{\alpha\beta}(z)$ too, are also defined on $\mathfrak{D}$ as integral operators.

Let us see what the preceding results imply as to the resolvent $\mathbf{R}(z)$. It is convenient to consider the operators

$$\mathbf{H}_\alpha=\mathbf{H}_0+\mathbf{V}_\alpha,\quad \alpha=23,\ 31,\ 12, \tag{3.25}$$

and their resolvents

$$\mathbf{R}_\alpha(z)=(\mathbf{H}_\alpha-z\mathbf{E})^{-1}. \tag{3.26}$$

It is easily verified that these resolvents are expressed through $\mathbf{T}_\alpha(z)$ as

$$\mathbf{R}_\alpha(z)=\mathbf{R}_0(z)-\mathbf{R}_0(z)\mathbf{T}_\alpha(z)\mathbf{R}_0(z), \tag{3.27}$$

so that

$$-\mathbf{R}_0(z)\mathbf{T}_\alpha(z)\mathbf{R}_0(z)=\mathbf{R}_\alpha(z)-\mathbf{R}_0(z). \tag{3.28}$$

The above analysis demonstrates

Theorem 3.1. *Let the functions* $v_\alpha(k)$ *satisfy conditions* A_{θ_0}, B_{μ_0}, R *with* $\theta_0>\frac{1}{2}$. *Then the resolvent* $\mathbf{R}(z)$ *of the operator* $\mathbf{H}$ *may be represented in the form*

$$\mathbf{R}(z)=\mathbf{R}_0(z)+\sum_\alpha(\mathbf{R}_\alpha(z)-\mathbf{R}_0(z))-\mathbf{R}_0(z)\sum_{\alpha,\beta}\mathbf{W}_{\alpha\beta}(z)\mathbf{R}_0(z), \tag{3.29}$$

where $\mathbf{W}_{\alpha\beta}(z)$ *are integral operators whose kernels satisfy estimates (3.24).*

Formula (3.29) follows from (3.5), (3.8), (3.14) and (3.28).

We conclude this section with a derivation of the properties possessed by the kernels $\mathcal{W}_{\alpha\beta}(k, p; k', p'; z)$ in virtue of the "self-adjointness" of the resolvent:

$$\mathbf{R}(z)=\mathbf{R}^*(\bar{z}) \tag{3.30}$$

and Hilbert's identity

$$\mathbf{R}(z_1)-\mathbf{R}(z_2)=(z_1-z_2)\mathbf{R}(z_1)\mathbf{R}(z_2). \tag{3.31}$$

Lemma 3.4. *The following relations are valid*

$$\mathcal{W}_{\alpha\beta}(k, p; k', p'; z)=\overline{\mathcal{W}_{\beta\alpha}(k', p'; k, p; \bar{z})}. \tag{3.32}$$

Proof. We note that the following relation holds, in view of (2.15), for any f and f' in $\mathfrak{D}$

$$(\mathbf{T}_\alpha(z)f, f')=(f, \mathbf{T}_\alpha(\bar{z})f'), \tag{3.33}$$

and on account of the symmetry of $\mathbf{V}_\alpha$ and property (3.30)

$$(\mathbf{M}_{\alpha\beta}(z)f, f')=(f, \mathbf{M}_{\beta\alpha}(\bar{z})f'). \tag{3.34}$$

Combining the last two relations, we obtain

$$(\mathbf{W}_{\alpha\beta}(z)f, f')=(f, \mathbf{W}_{\beta\alpha}(\bar{z})f'), \tag{3.35}$$

showing that the corresponding kernels satisfy (3.32).

Lemma 3.5. *The following relations are valid*

$$\mathcal{M}_{\alpha\beta}(k, p; k', p'; z_1)-\mathcal{M}_{\alpha\beta}(k, p; k', p'; z_2)=(z_2-z_1)\int\sum_\gamma\mathcal{M}_{\alpha\gamma}(k, p; k'', p''; z_1)\times$$

$$\times\left(\frac{k''^2}{2m}+\frac{p''^2}{2n}-z_1\right)^{-1}\left(\frac{k''^2}{2m}+\frac{p''^2}{2n}-z_2\right)^{-1}\sum_\gamma\mathcal{M}_{\gamma\beta}(k'', p''; k', p'; z_2)\,dk''dp''. \tag{3.36}$$

Proof. We apply the relations

$$\mathbf{V}_\alpha\mathbf{R}(z)=\sum_\gamma\mathbf{M}_{\alpha\gamma}(z)\mathbf{R}_0(z);\quad \mathbf{R}(z)\mathbf{V}_\beta=\mathbf{R}_0(z)\sum_\gamma\mathbf{M}_{\gamma\beta}(z), \tag{3.37}$$

which may be obtained by multiplying (3.5) on the left by $\mathbf{V}_\alpha$, (3.7) on the left by $\mathbf{R}_0(z)$, and making use of (3.8) and equations (3.2) and (3.9).

Multiplying Hilbert's identity by $\mathbf{V}_\alpha$ on the left and by $\mathbf{V}_\beta$ on the right, and applying (3.37), we obtain

$$\mathbf{M}_{\alpha\beta}(z_1)-\mathbf{M}_{\alpha\beta}(z_2)=(z_2-z_1)\sum_{\gamma}\mathbf{M}_{\alpha\gamma}(z_1)\mathbf{R}_0(z_1)\mathbf{R}_0(z_2)\sum_{\gamma}\mathbf{M}_{\gamma\beta}(z_2). \qquad (3.38)$$

Relation (3.36) is simply (3.38) written in terms of the kernels of the $\mathbf{M}_{\alpha\beta}(z)$. It should be kept in mind that these kernels have for $\alpha=\beta$ a δ-type singularity

$$\mathcal{M}_{\alpha\alpha}(k,\ p;\ k',\ p';\ z)=t_{\alpha}\left(k_{\alpha},\ k'_{\alpha},\ z-\frac{p_{\alpha}^{2}}{2n_{\alpha}}\right)\delta(p_{\alpha}-p'_{\alpha})+$$
$$+\mathcal{W}_{\alpha\alpha}(k,\ p;\ k',\ p';\ z),$$

where $\mathcal{W}_{\alpha\alpha}(k,\ p;\ k'\ p';\ z)$ is a smooth bounded kernel. This completes the proof.

§ 4. Investigation of the kernel $t(k,\ k',\ z)$

In this section we shall investigate the integral equation derived in § 2 for the kernel $t(k,\ k',\ z)$. We shall prove its solvability and obtain detailed estimates for the solution.

The equation in question has the form

$$t(k,\ k',\ z)=v(k-k')-\int v(k-k'')\left(\frac{k''^{2}}{2m}-z\right)^{-1}t(k'',\ k',\ z)\,dk''. \qquad (4.1)$$

This is an equation for the kernel $t(k,\ k',\ z)$, seen as a function of k, with k' and z serving as parameters. The dependence on k' is determined by the free term, and the dependence on z, by the kernel of the integral equation.

We will assume once and for all that the function $v(k)$ satisfies conditions A_{θ_0}, B_{μ_0} and R, with $\theta_0>\frac{1}{2}$, $\mu_0>\frac{1}{2}$.

The integral in equation (4.1) becomes singular when z lies on the real axis. In order to give such an integral a meaning for arbitrary real z, we impose a Hölder condition on all the functions appearing in the integrand. Let us now state a few definitions and results concerning Hölder functions and singular integrals involving them, of which frequent use will be made in the following.

The function $f(k_1,\ \ldots,\ k_n,\ z_1,\ \ldots,\ z_m)$ of the variables $k_1,\ \ldots,\ k_n$ and the complex variables $z_1,\ \ldots,\ z_m$ is said to fulfil Hölder's condition with the indices $\mu_1,\ \ldots,\ \mu_n,\ \nu_1,\ \ldots,\ \nu_m$ and the estimating function $M(k_1,\ \ldots,\ k_n,\ z_1,\ \ldots,\ z_m)$, where k_i, $i=1,\ \ldots,\ n$, range throughout three-dimensional space, and z_j, $j=1,\ \ldots,\ m$, throughout the region Π in the complex plane, if

$$|f(k_1,\ \ldots,\ k_n,\ z_1,\ \ldots,\ z_m)|\leqslant CM(k_1,\ \ldots,\ k_n,\ z_1,\ \ldots,\ z_m);$$
$$|f(k_1+h_1,\ \ldots,\ k_n+h_n,\ z_1+\Delta_1,\ \ldots,\ z_m+\Delta_m)-f(k_1,\ \ldots,\ k_n,\ z_1,\ \ldots,\ z_m)|\leqslant$$
$$\leqslant CM(k_1,\ldots,k_n,\ z_1,\ \ldots,\ z_m)\left[|h_1|^{\mu_1}+\ldots+|h_n|^{\mu_n}+|\Delta_1|^{\nu_1}+\ldots+|\Delta_m|^{\nu_m}\right],$$

where

$$|h_i|\leqslant 1,\ i=1,\ \ldots,\ n;\ |\Delta_j|\leqslant 1,\ j=1,\ \ldots,\ m,$$

for $z_j+\alpha\Delta_j\in\Pi$, $0\leqslant\alpha\leqslant 1$.

The following propositions are valid:

I. Let $f(k_1,\ \ldots,\ k_n,\ z_1,\ \ldots,\ z_m)$ be a Hölder function with the indices $\mu_1,\ \ldots,\ \mu_n$,

$\nu_1, \ldots, \nu_m$ and the estimating function $M(k_1, \ldots, k_n, z_1, \ldots, z_m)$. Then

$$f_1(k_1, \ldots, k_{n-2}, k, z_1, \ldots, z_m)=f(k_1, \ldots, k_n, z_1, \ldots, z_m)|_{k_{n-1}=k_n=k};$$
$$f_2(k_1, \ldots, k_n, z_1, \ldots, z_{m-2}, z)=f(k_1, \ldots, k_n, z_1, \ldots, z_m)|_{z_{m-1}=z_m=z}$$

too, are Hölder functions, having respectively the indices $\mu_1, \ldots, \mu_{n-2}, \mu, \nu_1, \ldots, \nu_m$, where $\mu=\min(\mu_{n-1}, \mu_n)$ and $\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_{m-2}, \nu$, where $\nu=\min(\nu_{m-1}, \nu_m)$, and the estimating functions $M(k_1, \ldots, k_{n-2}, k, k, z_1, \ldots, z_m)$ and $M(k_1, \ldots, k_n, z_1, \ldots, z_{m-2}, z, z)$.

II. Let $f(k_1, \ldots, k_n, z_1, \ldots, z_m)$ be the same as in I. Then we have the estimates

$$|f(k_1+h_1, k_2+h_2, \ldots)-f(k_1+h_1, k_2, \ldots)-f(k_1, k_2+h_2, \ldots)+$$
$$+f(k_1, k_2, \ldots)|\leqslant CM(k_1, \ldots, k_n, z_1, \ldots, z_m)|h_1|^{\mu_1\gamma}|h_2|^{\mu_2(1-\gamma)};$$
$$|f(k_1+h_1, \ldots, z_1+\Delta_1, \ldots)-f(k_1, \ldots, z_1+\Delta_1, \ldots)-$$
$$-f(k_1+h_1, \ldots, z_1, \ldots)+f(k_1, \ldots, z_1, \ldots)|\leqslant$$
$$\leqslant CM(k_1, \ldots, k_n, z_1, \ldots, z_m)|h_1|^{\mu_1\gamma}|\Delta_1|^{\nu_1(1-\gamma)};$$
$$|f(\ldots, z_1+\Delta_1, z_2+\Delta_2, \ldots)-f(\ldots, z_1+\Delta_1, z_2, \ldots)-$$
$$-f(\ldots, z_1, z_2+\Delta_2, \ldots)+f(\ldots, z_1, z_2, \ldots)|\leqslant$$
$$\leqslant CM(k_1, \ldots, k_n, z_1, \ldots, z_m)|\Delta_1|^{\nu_1\gamma}|\Delta_2|^{\nu_2(1-\gamma)}.$$

Here γ is an arbitrary number $0\leqslant\gamma\leqslant 1$. For brevity, we have indicated by dots those arguments of $f(k_1, \ldots, k_n, z_1, \ldots, z_m)$ which are held fixed.

III. Let $f(k_1, \ldots, k_n, z_1, \ldots, z_m)$ be as in I, and let

$$\int d\Omega_{k_n} M(k_1, \ldots, k_n, z_1, \ldots, z_m)\leqslant$$
$$\leqslant CN(k_1, \ldots, k_{n-1}, z_1, \ldots, z_m)(1+|k_n|)^{-1-\theta}.$$

Then

$$f_1(k_1, \ldots, k_{n-1}, z_1, \ldots, z_m, z)=\int\left(\frac{k_n^2}{2m}-z\right)^{-1} f(k_1, \ldots, k_n, z_1, \ldots, z_m)\,dk_n$$

too, is a Hölder function, having the indices $\mu_1', \ldots, \mu_{n-1}', \nu_1', \ldots, \nu_m', \nu$, where $\nu=\min\left(\frac{1}{2}, \mu_n\right)$ and each of the $\mu_1', \ldots, \mu_{n-1}', \nu_1', \ldots, \nu_m'$ may be taken as close as desired to, but smaller than the corresponding $\mu_1, \ldots, \mu_{n-1}, \nu_1, \ldots, \nu_m$, and the estimating function

$$M_1(k_1, \ldots, k_{n-1}, z_1, \ldots, z_m, z)=N(k_1, \ldots, k_{n-1}, z_1, \ldots, z_m)\times$$
$$\times(1+|z|)^{-\frac{\theta'}{2}}, \quad \theta'<\min(1, \theta),$$

where z ranges over the complex plane Π_0, slit along the positive real axis. The function $f_1(k_1, \ldots, k_{n-1}, z_1, \ldots, z_m, z)$ is everywhere differentiable with respect to z except at the positive real axis, and for $Re\, z<-1$ we have the estimate

$$\left|\frac{\partial}{\partial z} f_1(k_1, \ldots, k_{n-1}, z_1, \ldots, z_m, z)\right|\leqslant$$
$$\leqslant CN(k_1, \ldots, k_{n-1}, z_1, \ldots, z_m)(1+|z|)^{-1-\frac{\theta'}{2}}.$$

The constants in the estimates of $f_1(k_1, \ldots, k_{n-1}, z_1, \ldots, z_m, z)$ depend linearly on the constants of the estimates of $f(k_1, \ldots, k_n, z_1, \ldots, z_m)$.

All these are known results, in this or an equivalent formulation. For the sake of completeness they are proved in Appendix I. In the following we shall refer to proposition III as the lemma on singular integrals.

Now for the investigation of equation (4.1). The estimate of the free term as a function of k depends, generally speaking, on the parameter k'. This disadvantage disappears in the iterations of equation (4.1). We now apply the lemma on singular integrals to obtain estimates for these iterations. Let us begin with the second iteration

$$t_1(k, k', z) = -\int v(k-k'')\left(\frac{k''^2}{2m} - z\right)^{-1} v(k''-k')\,dk''. \tag{4.2}$$

In order to estimate the integrand, we make use of the elementary inequality

$$\begin{aligned}&(1+|k-q|)^{-\theta}(1+|q-k'|)^{-\theta} \leqslant \\ &\leqslant C[(1+|k-q|)^{-\theta} + (1+|q-k'|)^{-\theta}](1+|k-k'|)^{-\theta},\end{aligned} \tag{4.3}$$

which follows from the triangle inequality and the familiar

$$(a+b)^{\theta} \leqslant C(a^{\theta}+b^{\theta});\ a \geqslant 0;\ b \geqslant 0;\ 0 \leqslant \theta.$$

Before applying the lemma on singular integrals it is necessary to estimate the integral over angle variables

$$I(k, |q|, \theta) = \int d\Omega_q (1+|k-q|)^{-\theta}. \tag{4.4}$$

This integral will be shown in Appendix II to satisfy the estimate

$$I(k, |q|, \theta) \leqslant C(1+|k|)^{-\theta_1}(1+|q|)^{-\theta_2},\quad \theta_1+\theta_2=\theta<2. \tag{4.5}$$

This, together with (4.3) gives the following estimate for the integral of the estimating function of the integrand of (4.2) with respect to the angle variables

$$\begin{aligned}I(k, k', |k''|) &= \int d\Omega_{k''}(1+|k-k''|)^{-1-\theta_0}(1+|k''-k'|)^{-1-\theta_0} \leqslant \\ &\leqslant C(1+|k-k'|)^{-1-\theta_0}\int d\Omega_{k''}[(1+|k-k''|)^{-1-\theta_0}+(1+|k''-k'|)^{-1-\theta_0}] \leqslant \\ &\leqslant C(1+|k-k'|)^{-1-\theta_0}(1+|k''|)^{-1-\theta_0}.\end{aligned}$$

It is assumed here that $\theta_0 < 1$.

We may also proceed differently. Let $\delta < \theta_0$; we have

$$\begin{aligned}I(k, k', |k''|) &\leqslant C(1+|k-k'|)^{-1-\theta_0+\delta}\int d\Omega_k (1+|k-k''|)^{-\delta} \times \\ &\times [(1+|k-k''|)^{-1-\theta_0+\delta}+(1+|k''-k'|)^{-1-\theta_0+\delta}] \leqslant \\ &\leqslant C(1+|k|)^{-\delta}(1+|k-k'|)^{-1-\theta_0+\delta}(1+|k''|)^{-1-\theta_0+\delta}.\end{aligned}$$

The last step follows by applying Hölder's inequality with a suitable index.

We have thus obtained the result that the kernel $t_1(k, k', z)$ is a Hölder function with the indices $\mu, \mu, \frac{1}{2}$, where μ is as close as desired to μ_0, and with the estimating function

$$(1+|k-k'|)^{-1-\theta}(1+|z|)^{-\theta/2} \text{ or } (1+|k-k'|)^{-1-\theta+\delta}(1+|k|)^{-\delta},$$

where $\theta < 1$, $\theta < \theta_0$ and $\delta < \theta$.

A similar procedure may be applied to the subsequent iterations. We shall state the result only. Let us define the operators

$$\mathbf{t}_n(z) = (-1)^n \mathbf{v}\mathbf{r}_0\underbrace{(z)\mathbf{v}\cdots\mathbf{v}\mathbf{r}_0}_{\mathbf{r}_0\ n\text{ times}}(z)\mathbf{v},\quad n=2, 3, \ldots \tag{4.6}$$

and denote their kernels by $t_n(k, k', z)$. Then the following result holds:

Lemma 4.1. The following estimates are valid for the kernels $t_n(k, k', z)$ *for any finite* n

$$|t_n(k, k', z)| \leqslant C(1+|k-k'|)^{-1-\theta}(1+|z|)^{-\theta/2}; \tag{4.7}$$

$$|t_n(k+h, k'+h', z+\Delta)-t_n(k, k', z)| \leqslant$$

$$\leqslant C(1+|k-k'|)^{-1-\theta}(1+|z|)^{-\theta/2}[|h|^{\mu_1}+|h'|^{\mu_1}+|\Delta|^{\nu}]; \tag{4.8}$$

$$\theta<1;\ \theta<\theta_0;\ \mu_1<\mu_0;\ \nu<\frac{1}{2},$$

and the estimating function $(1+|k-k'|)^{-1-\theta}$ *in the right members of (4.7) and (4.8) may be replaced for* $n \geqslant n_0$, *where* n_0 *depends on* θ_0, *by the function* $(1+|k|)^{-\theta_1}(1+|k'|)^{-\theta_2}$ *where* θ_1 *and* θ_2 *are some nonnegative exponents which satisfy the relation* $\theta_1+\theta_2=1+\theta$. *All these estimates hold uniformly in all the* k, k' *and* $z \in \Pi_0$.

We introduce the class $\mathfrak{m}(\theta, \mu)$ of functions $f(k)$ satisfying the estimates

$$|f(k)| \leqslant C(1+|k|)^{-\theta};\quad |f(k+h)-f(k)| \leqslant C(1+|k|)^{-\theta}|h|^{\mu};$$

$$|h| \leqslant 1;\quad \theta>0;\quad 0<\mu<1. \tag{4.9}$$

This class will constitute a Banach space if the norm is defined as

$$\|f\|_{\theta,\mu}=\sup_{k,h}(1+|k|)^{\theta}\left[|f(k)|+\frac{|f(k+h)-f(k)|}{|h|^{\mu}}\right]. \tag{4.10}$$

We designate this space $\mathfrak{b}(\theta, \mu)$.

Let us remark that $v(k) \in \mathfrak{m}(1+\theta_0, \mu_0)$, and that the kernels $t_n(k, k', z)$ which considered as functions of k, belong for $n \geqslant n_0$ to $\mathfrak{m}(\theta_1, \mu_1)$ are uniformly estimated for $z \in \Pi_0$ by

$$\|t_n(\cdot, k', z)\|_{\theta_1, \mu_1} \leqslant C(1+|k'|)^{-\theta_2}(1+|z|)^{-\theta/2}; \tag{4.11}$$

$$\|t_n(\cdot, k'+h', z+\Delta)-t_n(\cdot, k', z)\|_{\theta_1, \mu_1} \leqslant$$

$$\leqslant C(1+|k'|)^{-\theta_2}(1+|z|)^{-\theta/2}[|h'|^{\mu_2}+|\Delta|^{\nu_2}]; \tag{4.12}$$

$$\theta_1+\theta_2<1+\theta_0;\quad \mu_1+\mu_2<\mu_0;\quad \nu_2<\mu_2/2\mu;\quad \theta<\theta_0.$$

The last estimate follows from our proposition II on Hölder functions.

We now consider the integral operator

$$\mathbf{a}(z)f(k)=\int v(k-k')\left(\frac{k'^2}{2m}-z\right)^{-1} f(k')dk'. \tag{4.13}$$

operating on $f(k) \in \mathfrak{m}(\theta, \mu)$. Repeating the procedure which we have used to estimate the iterations of equation (4.1), we may prove the following:

Lemma 4.2. Let $f(k) \in \mathfrak{m}(\theta, \mu)$ *and* $g(k, z)=\mathbf{a}(z)f(k)$, *where* $\theta<1+\theta_0$ *and* $\mu<\mu_0$. *Then* $g(k, z)$ *satisfies the estimates*

$$|g(k, z)| \leqslant C\|f\|_{\theta,\mu}(1+|k|)^{-\theta'}(1+|z|)^{-\delta}; \tag{4.14}$$

$$|g(k+h, z+\Delta)-g(k, z)| \leqslant$$

$$\leqslant C\|f\|_{\theta,\mu}(1+|k|)^{-\theta'}(1+|z|)^{-\delta}[|h|^{\mu'}+|\Delta|^{\nu'}]; \tag{4.15}$$

$$\theta' \leqslant 1+\theta_0;\quad \theta'<\theta_0+\theta;\quad \delta<\frac{\theta_0+\theta-\theta'}{2},$$

where $\nu'=\min\left(\frac{1}{2}, \mu\right)$ *and* μ' *does not depend on* μ *and may be taken as close as desired to* μ_0.

Thus, the operator $\mathbf{a}(z)$ transforms functions belonging to the class $\mathfrak{m}(\theta, \mu)$, with $\theta < 1+\theta_0$, $\mu < \mu_0$, into functions of the class $\mathfrak{m}(\theta', \mu')$, where we may take $\mu' > \mu$, $\theta' > \theta$. We may note that the sequence of functions $f_n(k)$, which is bounded in the norm of $\mathfrak{b}(\theta', \mu')$, must be compact in any $\mathfrak{b}(\theta, \mu)$, $\theta < \theta'$, $\mu < \mu'$. This criterion of compactness, and the estimates (4.14) and (4.15) imply the validity of

Lemma 4.3. For any $z \in \Pi_0$, *the integral operator* $\mathbf{a}(z)$ *determines in* $\mathfrak{b}(\theta, \mu)$, $\theta < 1+\theta_0$, $\mu < \mu_0$ *a completely continuous operator,*

$$\|\mathbf{a}(z)\|_{\theta,\mu} \leqslant C(1+|z|)^{-\delta}; \tag{4.16}$$

$$\|\mathbf{a}(z+\Delta)-\mathbf{a}(z)\|_{\theta,\mu} \leqslant C(1+|z|)^{-\delta}|\Delta|^{\nu}, \tag{4.17}$$

while δ *may be arbitrarily close to* $\frac{\theta_0}{2}$, *and* $\nu < \frac{\mu_0 - \mu}{2\mu_0}$.

Consider now, instead of $t(k, k', z)$, the kernel $\tilde{t}(k, k', z)$, obtained from the first by subtraction of the first $n_0 - 1$ iterations of equation (4.1):

$$\tilde{t}(k, k', z) = t(k, k', z) - t_1(k, k', z) - \ldots - t_{n_0-1}(k, k', z).$$

The equation for $\tilde{t}(k, k', z)$ is

$$\tilde{t}(k, k', z) = t_{n_0}(k, k', z) - \int v(k-k'')\left(\frac{k''^2}{2m} - z\right)^{-1} \tilde{t}(k'', k', z)\, dk'', \tag{4.18}$$

and, as mentioned before, the free term belongs to $\mathfrak{m}(\theta_1, \mu_1)$, $\theta_1 < 1+\theta_0$, $\mu_1 < \mu_0$. We are looking for a solution $\tilde{t}(k, k', z)$ out of the class of functions, which for fixed k' and z belong to $\mathfrak{m}(\theta_1, \mu_1)$ as functions of k. On denoting by $f_0(k', z)$ and $f(k', z)$ the elements induced by these kernels in $\mathfrak{b}(\theta_1, \mu_1)$, (4.18) can be written

$$f(k', z) = f_0(k', z) - \mathbf{a}(z) f(k', z) \tag{4.19}$$

We gather from Lemma 4.3 that the Fredholm alternative applies to this equation, so that our next problem is to study the homogeneous equation

$$\varphi + \mathbf{a}(z)\varphi = 0 \tag{4.20}$$

in the function class $\mathfrak{m}(\theta, \mu)$, $\theta < 1+\theta_0$, $\mu < \mu_0$.

Lemma 4.4. Let $\varphi(k)$ *be a solution of equation (4.20) belonging to* $\mathfrak{m}(\theta, \mu)$, $\theta < 1+\theta_0$, $\mu < \mu_0$. *Then* $\varphi(k) \in \mathfrak{m}(1+\theta_0, \mu')$, *where* μ' *may be taken as close as desired to* μ_0.

The proof follows by repeated application of the estimates of Lemma 4.2.

Lemma 4.5. Let $\operatorname{Im} z \neq 0$. *Then equation (4.20) has no nontrivial solutions in* $\mathfrak{m}(\theta, \mu)$.

Proof. Let (4.20) have a solution, $\varphi_0(k)$, in $\mathfrak{m}(\theta, \mu)$ for $z = z_0$. By Lemma 4.4 $\varphi_0(k) \in \mathfrak{m}(1+\theta_0, \mu)$, so that $\varphi_0(k) \in \mathfrak{h}$ and $\psi_0(k) = \left(\frac{k^2}{2m} - z_0\right)^{-1} \cdot \varphi_0(k) \in \mathfrak{b}$. Equation (4.20) then becomes, in terms of $\psi_0(k)$,

$$\left(\frac{k^2}{2m} - z_0\right)\psi_0(k) + \int v(k-k')\psi_0(k')\, dk' = 0$$

or

$$\mathbf{h}\psi_0 = z_0 \psi_0.$$

Since $\mathbf{h}$ is self-adjoint it follows that $\psi_0 \equiv 0$, and hence $\varphi_0(k) = 0$. This completes the proof.

Lemma 4.6. Equation (4.20) has no nontrivial solutions for sufficiently large $|z|$.

The proof follows from the fact that the norm of $\mathbf{a}(z)$ is shown by the estimate (4.16) to be smaller than 1 for sufficiently large $|z|$.

Lemma 4.7. Let $\varphi(k) \in \mathfrak{m}(\theta, \mu)$ *be a solution of (4.20) for* $z=\omega^2+i0$ *or* $z=\omega^2-i0$. *Then*

$$\varphi(k)\Big|_{\frac{k^2}{2m}=\omega^2}=0. \tag{4.21}$$

Proof. Consider the case $z=\omega^2+i0$. The element

$$g(\varepsilon)=[\mathbf{a}(\omega^2+i\varepsilon)-\mathbf{a}(\omega^2+i0)]\varphi,$$

constructed from a solution $\varphi(k)$ of equation (4.20) for $z=\omega^2+i0$ must vanish in the norm of $\mathfrak{b}(\theta_0, \mu)$ for $\varepsilon\to 0$, on account of the continuity of the operator $\mathbf{a}(z)$. The functions $\varphi(k)$ and $g(k, \varepsilon)$ belong to $\mathfrak{h}$, and the equation

$$\varphi+\mathbf{a}(\omega^2+i\varepsilon)\varphi=g(\varepsilon)$$

may be written

$$\varphi+\mathbf{v}\mathbf{r}_0(\omega^2+i\varepsilon)\varphi=g(\varepsilon). \tag{4.22}$$

We form the scalar product of (4.22) and $\mathbf{r}_0(\omega^2+i\varepsilon)\varphi$, and obtain in virtue of the symmetry of $\mathbf{v}$ in $\mathfrak{b}$,

$$\operatorname{Im}(\mathbf{r}_0(\omega^2+i\varepsilon)\varphi,\ \varphi)=\operatorname{Im}(\mathbf{r}_0(\omega^2+i\varepsilon)\varphi,\ g(\varepsilon)). \tag{4.23}$$

The integrals

$$\int\varphi(k)\left(\frac{k^2}{2m}-(\omega^2\pm i\varepsilon)\right)^{-1}g(k, \varepsilon)\,dk,$$

appearing on the right-hand side of (4.23) vanish in the limit for $\varepsilon\to 0$, so that (4.23) may be written

$$([\mathbf{r}_0(\omega^2+i\varepsilon)-\mathbf{r}_0(\omega^2-i\varepsilon)]\varphi,\ \varphi)=o(1)$$

or

$$\int\varphi(k)\left[\frac{1}{\frac{k^2}{2m}-\omega^2-i\varepsilon}-\frac{1}{\frac{k^2}{2m}-\omega^2+i\varepsilon}\right]\overline{\varphi(k)}\,dk=o(1).$$

Passing to the limit for $\varepsilon\to 0$, we obtain

$$\int|\varphi(k)|^2\delta\left(\frac{k^2}{2m}-\omega^2\right)dk=0,$$

which implies (4.21).

The case $z=\omega^2-i0$ is treated similarly. Note that we have proved that equation (4.20) has the same solutions for $z=\omega^2+i0$ and $z=\omega^2-i0$. Indeed, for any solution of (4.20) with $z=\omega^2+i0$ we have

$$(\mathbf{a}(\omega^2+i0)-\mathbf{a}(\omega^2-i0))\varphi(k)=2\pi i\int v(k-k')\delta\left(\frac{k'^2}{2m}-\omega^2\right)\varphi(k')\,dk'=0,$$

which shows that $\varphi(k)$ also satisfies (4.20) with $z=\omega^2-i0$. This completes the proof.

The points λ on the real axis at which equation (4.20) has nontrivial solutions, will be called the singular points of the operator $\mathbf{a}(z)$.

Lemma 4.8. Any singular point $\lambda\neq 0$ *of the operator* $\mathbf{a}(z)$ *belongs to the discrete spectrum of* $\mathbf{h}$.

Proof. Consider first the case $\lambda=\omega^2>0$. We construct from the solution

$\varphi(k)$ of (4.20) for $z=\omega^2$ the function

$$\psi(k)=\varphi(k)\left(\frac{k^2}{2m}-\omega^2\right)^{-1}.$$

Let us show that $\psi(k)\in\mathfrak{h}$. The difficulty arises only in the neighborhood of the surface $\frac{k^2}{2m}=\omega^2$, where the denominator of $\psi(k)$ vanishes. However, by Lemma 4.7 we have for $\frac{k^2}{2m}\approx\omega^2$

$$|\psi(k)|=\left|\varphi(k)-\varphi(k)\right|_{\frac{k^2}{2m}=\omega^2}\left|\frac{k^2}{2m}-\omega^2\right|^{-1}\leqslant C\|\varphi\|_{\theta,\mu}\left|\frac{k^2}{2m}-\omega^2\right|^{-1+\mu}.$$

Since $\mu_0>\frac{1}{2}$, we may in view of Lemma 4.4 assume that $\mu>\frac{1}{2}$, too, so that $\psi(k)$ is square-integrable in the neighborhood of the singular surface. Equation (4.20) becomes in terms of $\psi(k)$

$$\left(\frac{k^2}{2m}-\omega^2\right)\psi(k)=-\int v(k-k')\psi(k')\,dk',$$

so that $\psi(k)$ is an eigenfunction of $\mathbf{h}$, and ω^2 is the corresponding eigenvalue.

The case $\lambda>0$, which is treated analogously, is simpler since the difficulty associated with the singular denominator does not arise here. This completes the proof.

These considerations become meaningless for $\lambda=0$. Examples show that this point may be singular as well as regular, and if it is singular it need not belong to the discrete septrum of $\mathbf{h}$. The point $\lambda=0$ is singular only in exceptional cases. Namely, if $\lambda=0$ is a singular point of the operator $\mathbf{a}(z)$ with the potential $v(k)$, then it will also be a singular point of the operator $\mathbf{a}_\varepsilon(z)$, where $v(k)$ is replaced by $(1+\varepsilon)v(k)$ for any sufficiently small ε. In fact, $\mathbf{a}_\varepsilon(z)=(1+\varepsilon)\mathbf{a}(z)$ and the point $1+\varepsilon$, for sufficiently small ε, does not belong to the spectrum of the completely continuous operator $\mathbf{a}(0)$.

The discrete spectrum of $\mathbf{h}$ was studied by many authors, and the conditions restricting the potential $\mathbf{v}$ were formulated in the configuration representation. Thus, Kato /5/ has proved that if $v(x)$ is a quite arbitrary function within a sphere of finite radius, but outside this sphere

$$|v(x)|\leqslant C(1+|x|)^{-\alpha} \tag{4.24}$$

where $\alpha>1$, then the operator $\mathbf{h}$ has no positive discrete spectrum. There exist numerous sufficient conditions for the finiteness of the negative spectrum. For example, the author /6/ has shown that if $\alpha>2$ in (4.24) then the nonpositive spectrum of $\mathbf{h}$ consists of a finite number of eigenvalues with finite multiplicities. Quite general conditions for finiteness of the nonpositive spectrum, which are almost necessary, are given by M. Sh. Birman /7/.

Our present method is ill-adapted to the study of the discrete spectrum. All that could be proved under the assumptions $\mathbf{A}_\theta$ and $\mathbf{B}_\mu$ is that the singular points of $\mathbf{a}(z)$ constitute a denumerable closed set, lie in a finite interval, and have at most one limit point, at $\lambda=0$. In order to apply the above results, it is necessary to impose on the Fourier transform of the potential $v(k)$ conditions of the type (4.24). The corresponding conditions may be referred to the function $v(k)$ itself, but these are known to be too broad. We shall therefore simply assume that the following condition is satisfied in addition to $\mathbf{A}_\theta$, $\mathbf{B}_\mu$ and $\mathbf{R}$.

Condition C. The discrete spectrum of $\mathbf{h}$ *consists of a single simple nonpositive eigenvalue. The point* $\lambda=0$ *is not an eigenvalue of* $\mathbf{a}(z)$.

The first restriction is imposed only in order to simplify the formulas. All the methods applied below and also all the results can be readily extended to the case of a finite number of negative eigenvalues having finite multiplicities. The second restriction of condition C is more essential; indeed, our subsequent treatment of the resolvent of the three-body energy operator can not be applied without modifications to the case when $\lambda=0$ is a singular point of at least one of the three operators of type $\mathbf{h}$ with the potentials $v_\alpha(k)$ and masses $m_\alpha(\alpha=23,\ 31,\ 12)$ standing for $v(k)$ and m. However, as already mentioned above, this second restriction is fulfilled as a rule.

We denote the eigenvalue of $\mathbf{h}$ by $-\varkappa^2$ and the corresponding eigenfunction by $\psi(k)$.

Lemma 4.9. The function $\psi(k)$ *may be represented in the form*

$$\psi(k)=\frac{\varphi(k)}{\frac{k^2}{2m}+\varkappa^2}, \tag{4.25}$$

where $\varphi(k)$ *belongs to* $\mathfrak{m}(1+\theta_0,\ \mu_0)$ *and satisfies equation (4.20) for* $z=-\varkappa^2$.

Proof. By definition, $\psi(k)$ belongs to $\mathfrak{h}$ and satisfies the equation

$$\frac{k^2}{2m}\psi(k)+\int v(k-k')\psi(k')\,dk'=-\varkappa^2\psi(k). \tag{4.26}$$

Substituting in (4.26) $\varphi(k)=\left(\frac{k^2}{2m}+\varkappa^2\right)\psi(k)$, we obtain

$$\varphi(k)=-\int v(k-k')\psi(k')\,dk'. \tag{4.27}$$

We have for $\varphi(k)$ the estimate

$$|\varphi(k)|\leqslant C\Big(\int(1+k'^2)^{-2}(1+|k-k'|)^{-(2+2\theta_0)}dk'\Big)^2\times$$
$$\times\Big(\int(1+k'^2)^2|\psi(k')|^2dk'\Big)^{1/2}\leqslant C(1+|k|)^{-1-\theta};$$
$$\theta<\frac{1}{2}.$$

Here the integral of the type

$$I(a)=\int\frac{dq}{(1+|q|)^\alpha}\cdot\frac{1}{(1+|q-a|)^\beta} \tag{4.28}$$

was estimated by

$$I(a)\leqslant C(1+|a|)^{-(\alpha-\beta-3)};\quad \alpha+\beta>3;\quad \alpha<3;\ \beta<3. \tag{4.29}$$

This estimate is proved in Appendix II. One may similarly estimate the Hölder difference for the function $\varphi(k)$ with index μ_0. We are led to the conclusion that $\varphi(k)\in\mathfrak{m}(1+\theta,\ \mu_0)$, $\theta<\frac{1}{2}$. Relation (4.27) assumes in terms of $\varphi(k)$ the form

$$\varphi(k)=-\int v(k-k')\left(\frac{k'^2}{2m}+\varkappa^2\right)^{-1}\varphi(k')\,dk',$$

i. e., $\varphi(k)$ is a solution of equation (4.20) for $z=-\varkappa^2$. We may conclude from Lemma 4.4 that $\varphi(k)\in\mathfrak{m}(1+\theta_0,\ \mu_0)$.

This completes the proof.

We have now completed the discussion of the set of singular points of $\mathbf{a}(z)$ and may go back to study equation (4.19), i. e., to investigate the operator $(\mathbf{e}+\mathbf{a}(z))^{-1}$.

Lemma 4.10. For any $z \in \Pi_0$ *except* $z = -\varkappa^2$, *the operator* $(\mathbf{e} + \mathbf{a}(z))^{-1}$ *has an inverse of the form*

$$(\mathbf{e} + \mathbf{a}(z))^{-1} = \mathbf{e} + \mathbf{b}(z), \tag{4.30}$$

where

$$\|\mathbf{b}(z)\|_{\theta,\mu} \leqslant C(1 + |z|)^{-\delta} \tag{4.31}$$

and

$$\|\mathbf{b}(z+\Delta) - \mathbf{b}(z)\|_{\theta,\mu} \leqslant C(1+|z|)^{-\delta}|\Delta|^{\nu}; \quad \delta < \frac{\theta_0}{2}; \quad \nu < \frac{\mu_0 - \mu}{2\mu_0}. \tag{4.32}$$

Proof. Fredholm's alternative and the absence of nontrivial solutions of the homogeneous equation imply that $\mathbf{e} + \mathbf{a}(z)$ has for any fixed $z \neq -\varkappa^2$ an inverse in $\mathfrak{b}(\theta, \mu)$. The estimate (4.31), which holds uniformly in $|z|$, follows from estimate (4.16) of $\mathbf{a}(z)$. (4.32) follows from the Hölder continuity of $\mathbf{a}(z)$. This completes the proof.

Applying the last lemma to equation (4.19) and keeping in mind the connection between this equation and (4.1), we deduce the following result:

Theorem 4.1. Let the conditions A_{θ_0}, B_{μ_0}, R *and* C *be fulfilled, with* $\theta_0 > \frac{1}{2}$, $\mu_0 > \frac{1}{2}$. *Then equation (4.1) has for any* $z \in \Pi_0$ *except* $z = -\varkappa^2$, *a unique solution which satisfies the estimates*

$$|t(k, k', z)| \leqslant C(1 + |k - k'|)^{-(1+\theta)}; \tag{4.33}$$

$$|t(k+h, k'+h', z+\Delta) - t(k, k', z)| \leqslant$$

$$\leqslant C(1 + |k-k'|)^{-1-\theta}[|h|^{\mu} + |h'|^{\mu} + |\Delta|^{\nu}]; \tag{4.34}$$

$$|t(k, k', z) - v(k-k')| \leqslant C(1 + |k-k'|)^{-1-\theta}(1+|z|)^{-\theta/2}, \tag{4.35}$$

where θ, μ, ν *may be chosen as close as desired to* $\min(1, \theta_0)$, μ_0 *and* $\frac{1}{2}$, *respectively.*

We have here made use of the following result. If

$$|f(k, k')| \leqslant C(1+|k|)^{-\theta_1}(1+|k'|)^{-\theta_2} \tag{4.36}$$

for any θ_1, θ_2 such that $\theta_1 + \theta_2 = \theta$, then

$$|f(k, k')| \leqslant C(1 + |k - k'|)^{-\theta}. \tag{4.37}$$

To verify this, let $|k| \leqslant |k'|$. Then

$$|k - k'| \leqslant |k| + |k'| \leqslant 2|k'|,$$

which gives (4.37) if $\theta_2 = \theta$ in (4.36). The case $|k| \geqslant |k'|$ is similar.

We now take up the investigation of the behavior of the kernel $t(k, k', z)$ in the neighborhood of the singular point $z = -\varkappa^2 < 0$, which by assumption belongs to the discrete spectrum of $\mathbf{h}$. It is easily verified that this is an isolated spectrum point. Indeed, for $z \neq -\varkappa^2$ we know that $t(k, k', z)$ satisfies the estimate (4.33), and the resolvent $\mathbf{r}_0(z)$, and thus also $\mathbf{r}(z)$, are bounded for any z in the neighborhood of $z = -\varkappa^2$.

Lemma 4.11. In the neighborhood of the singular point $z = -\varkappa^2$ *the operator* $\mathbf{t}(z)$ *admits the representation*

$$\mathbf{t}(z) = (z + \varkappa^2)^{-1}\mathbf{c} + \hat{\mathbf{t}}(z), \tag{4.38}$$

where $\mathbf{c}$ *and* $\hat{\mathbf{t}}(z)$ *are defined on* $\mathfrak{b}$ *for all the* z *in question, and* $\mathbf{c}$ *is symmetric.*

Proof. We use the following representation of $\mathbf{r}(z)$ in the neighborhood of the isolated spectrum point of $\mathbf{h}$

$$\mathbf{r}(z) = -(\varkappa^2 + z)^{-1}\mathbf{p} + \hat{\mathbf{r}}(z). \tag{4.39}$$

Here $\mathbf{p}$ is a projection operator on the subspace corresponding to $\lambda=-\varkappa^2$, and $\hat{\mathbf{r}}(z)$ is an operator which is bounded for $z\approx-\varkappa^2$ and analytic in z. The ranges of $\mathbf{p}$ and $\hat{\mathbf{r}}(z)$ lie in $\mathfrak{h}$. Formula (4.38) follows from the expression (2.8) for $\mathbf{t}(z)$ in terms of $\mathbf{r}(z)$, with

$$\mathbf{c}=-\mathbf{vpv};\quad \hat{\mathbf{t}}(z)=\mathbf{v}-\mathbf{v}\hat{\mathbf{r}}(z)\mathbf{v}, \tag{4.40}$$

and these operators possess the required properties. This completes the proof.

Lemma 4.12. The operator $\mathbf{c}$ *is an integral operator, and its kernel may be represented as*

$$c(k,k')=\varphi(k)\overline{\varphi(k')}, \tag{4.41}$$

where $\varphi(k)$ *is a solution of equation (4.20) for* $z=-\varkappa^2$, *normalized by the condition*

$$\int\varphi(k)\left(\frac{k^2}{2m}+\varkappa^2\right)^{-2}\overline{\varphi(k)}\,dk=1. \tag{4.42}$$

Proof. The projection operator $\mathbf{p}$ is an integral operator with the kernel

$$p(k,k')=\psi(k)\overline{\psi(k')}, \tag{4.43}$$

where $\psi(k)$ is the normalized-to-unity eigenfunction of $\mathbf{h}$ that belongs to the eigenvalue $-\varkappa^2$. The lemma follows from (4.27), (4.40) and (4.43).

Lemma 4.13. The kernel $t(k,k',z)$ *has the following representation*

$$t(k,k',z)=\frac{\varphi(k)\overline{\varphi(k')}}{z+\varkappa^2}+v(k-k')+$$
$$+\int t\left(k,q,\frac{q^2}{2m}\pm i0\right)\left(\frac{q^2}{2m}-z\right)^{-1}t\left(q,k',\frac{q^2}{2m}\mp i0\right)dq. \tag{4.44}$$

Proof. We note that the kernel $t(k,k',z)$ is for fixed k and k' an analytic function of z throughout the complex plane except at $z=-\varkappa^2$ and at the slit along the positive real axis, where it has continuous limiting values. Actually, passing to the limit in (2.17) for $z_1\to z_2$, we obtain

$$\frac{d}{dz}t(k,k',z)=-\int t(k,q,z)\left(\frac{q^2}{2m}-z\right)^{-2}t(q,k',z)\,dq, \tag{4.45}$$

and the integral on the right-hand side is easily estimated for any admissible z by means of the obtained estimates. In virtue of this analyticity, the kernel $t(k,k',z)$ may be written

$$t(k,k',z)=\frac{1}{2\pi i}\oint_{\gamma}\frac{t(k,k',s)}{s-t}\,ds,$$

where the contour γ consists of a circle of radius ε around the point $z=-\varkappa^2$ and a contour which runs along the slit at a distance ε on both sides and widens into a circle of radius R around the origin, which closes it. The integral along the small circle reduces in the limit for $\varepsilon\to0$ to

$$\lim_{\varepsilon\to0}\frac{1}{2\pi i}\oint_{|s+\varkappa^2|=\varepsilon}\frac{t(k,k',s)}{s-z}\,ds=\frac{\varphi(k)\overline{\varphi(k')}}{z+\varkappa^2}. \tag{4.46}$$

Since all we know about the behavior of $t(k,k',z)$ in the neighborhood of $z=-\varkappa^2$ is the assertion of Lemma 4.11, we proceed as follows in order to prove (4.46). We take two smooth finite functions $f(k)$ and $f'(k)$ and consider the function

$$t(s)=\int\overline{f'(k)}\,t(k,k',s)f(k')\,dk\,dk'=(\mathbf{t}(s)f,f').$$

Since on the functions of $\mathfrak{d}$ the operator $\tilde{\mathfrak{t}}(z)$ is uniformly bounded in z,

$$\lim_{\varepsilon\to 0} \oint_{|s+\varkappa^2|=\varepsilon} \frac{t(s)}{s-z}ds = \lim_{\varepsilon\to 0}\left\{ \oint_{|s+\varkappa^2|=\varepsilon} \left[\frac{(\mathfrak{c}f,\ f')}{(s+\varkappa^2)(s-z)} + \frac{(\hat{\mathfrak{t}}(s)f,\ f')}{s-z}\right]ds\right\} =$$
$$= 2\pi i\,\frac{(\mathfrak{c}f,\ f')}{z+\varkappa^2} = 2\pi i\,\frac{1}{z+\varkappa^2}\int f(k)\overline{\varphi(k)}\,dk\int \overline{f'(k)}\,\varphi(k)\,dk,$$

which entails (4.46) on account of the arbitrariness of the functions $f(k)$ and $f'(k)$. Let us now examine the contribution of the second contour. Using the asymptotic estimate (4.35), we find for the integral along the large circle, in the limit for $R\to\infty$, the value $v(k-k')$. The integral along the lines running along the slit is evaluated by letting them approach the slit. The expression $t(k,\ k',\ s+i0)-t(k,\ k',\ s-i0)$ which then appears may be represented by means of (2.17), substituting there $z_1=s+i\varepsilon$ and $z_2=s-i\varepsilon$ and passing to the limit for $\varepsilon\to 0$,

$$t(k,\ k',\ s+i0)-t(k,\ k',\ s-i0)=$$
$$=\lim_{\varepsilon\to 0}\int t(k,\ q,\ s+i\varepsilon)\frac{2i\,|\varepsilon|}{\left(\frac{q^2}{2m}-s\right)^2+\varepsilon^2}\,t(q,\ k',\ s-i\varepsilon)\,dq=$$
$$=2\pi i\int t\left(k,\ q,\ \frac{q^2}{2m}\pm i0\right)\delta\left(\frac{q^2}{2m}-s\right)t\left(q,\ k',\ \frac{q^2}{2m}\mp i0\right)dq.$$

Collecting the various contributions, we obtain (4.44). This completes the proof.

Lemma 4.14. The functions

$$t^{(\pm)}(k,\ k')=t\left(k,\ k',\ \frac{k'^2}{2m}\pm i0\right) \tag{4.47}$$

satisfy the estimates

$$|t^{(\pm)}(k,\ k')|\leqslant C(1+|k-k'|)^{-1-\theta};$$
$$|t^{(\pm)}(k+h,\ k'+h')-t^{(\pm)}(k,\ k')|\leqslant$$
$$\leqslant C(1+|k-k'|)^{-1-\theta}[|h|^{\nu}+|h'|^{\nu}], \tag{4.48}$$

where ν *can be taken as close as desired to* $\frac{1}{2}$.

The proof requires only a Hölder-type estimation with respect to k'. We have, for example,

$$\left|t\left(k,\ k',\ \frac{(q+h)^2}{2m}+i0\right)-t\left(k,\ k',\ \frac{q^2}{2m}+i0\right)\right|\leqslant$$
$$\leqslant C(1+|k-k'|)^{-1-\theta}(1+q^2)^{-\theta/2}(1+|q|)^{1/2}|h|^{\nu}. \tag{4.49}$$

We may take $\theta>\frac{1}{2}$ in (4.49); the increasing factor $(1+|q|)^{1/2}$ is offset by the denominator. Substituting $k'=q$ in (4.49), we obtain the estimates (4.48).

We now possess all we need to prove the following theorem:

Theorem 4.2. Let conditions A_{θ_0}, B_{μ_0}, R *and* C *be fulfilled with* $\theta_0>\frac{1}{2}$ *and* $\mu_0>\frac{1}{2}$. *Then the kernel* $t(k,\ k',\ z)$ *may be represented in the form*

$$t(k,\ k',\ z)=\frac{\varphi(k)\varphi(k')}{z+\varkappa^2}+\hat{t}(k,\ k',\ z), \tag{4.50}$$

where $\varphi(k)\in\mathfrak{m}(1+\theta_0,\ \mu_0)$, *and* $\hat{t}(k,\ k',\ z)$ *satisfies the following estimates uniformly in* Π_0

$$|\hat{t}(k,\ k',\ z)\leqslant C(1+|k-k'|)^{-1-\theta}; \tag{4.51}$$
$$|\hat{t}(k+h,\ k'+h',\ z+\Delta)-\hat{t}(k,\ k',\ z)|\leqslant$$
$$\leqslant C(1+|k-k'|)^{-1-\theta}[|h|^{\mu}+|h'|^{\mu}+|\Delta|^{\nu}], \tag{4.52}$$

and θ, μ, ν *may be made as close as desired to* θ_0, μ_0, $\frac{1}{2}$, *respectively. Further,*

$$\left|\frac{\partial}{\partial z}\hat{t}(k, k', z)\right| \leqslant C(1+|k-k'|)^{-1-\theta}(1+|z|)^{-1} \qquad (4.53)$$

for $\mathrm{Re}\, z < -1$.

Proof. We apply the representation (4.44) derived in Lemma 4.13 for the kernel $t(k, k', z)$, and let $\hat{t}(k, k', z)$ stand for the last two terms of (4.44). It only remains to justify the estimate for the integral in (4.44). We may write the numerator of the integrand in the form

$$t\left(k, q, \frac{q^2}{2m}+i0\right)\overline{t\left(k', q, \frac{q^2}{2m}+i0\right)},$$

by virtue of the symmetry condition (2.15) for the kernel $t(k, k', z)$. The estimate now follows from the lemma on singular integrals and Lemma 4.14, exactly as the estimates of the iterations of equation (4.1).

§ 5. General treatment of the kernels $\mathcal{W}_{\alpha\beta}(k, p; k', p'; z)$

In this and the next two sections we will examine the behavior of the kernels $\mathcal{W}_{\alpha\beta}(k, p; k', p'; z)$ of the operators $\mathbf{W}_{\alpha\beta}(z)$, defined in § 3. Our basic tool is the system of equations (3.15), derived for these operators in § 3.

The investigation is conducted along the same lines as in our treatment of equation (4.1) in § 4, that is, our task is divided as follows.

(i) Obtaining suitable estimates for the iterations of the system of equations (3.20).

(ii) Finding a function space in which these iterations are contained, at least from a certain order onwards, and such that the system of equations (3.20) can be written as an integral equation of the second kind in terms of its elements and operators.

(iii) Investigating the associated homogeneous equation.

In the present section we for the most part only state the results, relegating their proofs to the next two sections.

It is assumed everywhere that conditions A_{θ_0}, B_{μ_0}, R and C are fulfilled for the three potentials $v_\alpha(k)$, $\alpha=23, 31, 12$. We denote by $-\varkappa_\alpha^2$ and $\psi_\alpha(k)$ the discrete eigenvalue and the normalized eigenfunction of the $\mathbf{h}$-type operator with m_α and $v_\alpha(k)$ standing for the mass m and the potential $v(k)$. The kernel of the operator $\mathbf{t}_\alpha(z)$ is noted by $t_\alpha(k, k', z)$, and $\varphi_\alpha(k)=\left(\frac{k^2}{2m_\alpha}+\varkappa_\alpha^2\right)\psi_\alpha(k)$.

We start with the description of the iterations of equation (3.15). These, we recall, may all be expressed by means of the operators $\mathbf{Q}^{(n)}_{\gamma_1,\dots,\gamma_{n+1}}(z)$, introduced in § 3. There we have also written down the following expression for the kernel of the operator $\mathbf{Q}^{(1)}_{23,\,31}(z)$ which appears in the first iteration (cf. (3.19)):

$$\mathcal{Q}^{(1)}_{23,\,31}(k, p; k', p'; z)=t_{23}\left(k_{23}, -p'_2-\frac{m_2}{m_2+m_3}p_1, z-\frac{p_1^2}{2n_1}\right)\times$$
$$\times\left(\frac{p_1^2}{2m_{31}}+\frac{(p_1, p'_2)}{m_3}+\frac{p_2'^2}{2m_{23}}-z\right)^{-1} t_{31}\left(p_1+\frac{m_1}{m_1+m_3}p'_2, k'_{31}, z-\frac{p_2'^2}{2n_2}\right). \qquad (5.1)$$

This kernel is seen to behave properly for complex z with $\mathrm{Im}\, z\neq 0$ and to possess several singularities for $\mathrm{Im}\, z\to 0$, some of which have their origin in the singularities of the kernels $t_{23}(k, k', z)$ and $t_{31}(k, k', z)$ involved in this expression; in (5.1) we have, for example, a term of the type

$$\frac{\varphi_{23}(k_{23})}{z+\varkappa_{23}^2-\frac{p_1^2}{2n_1}}\mathcal{G}_{23,\,31}^{(1)}(p_1;\ k',\ p';\ z), \tag{5.2}$$

where $\mathcal{G}_{23,\,31}^{(1)}(p_1;\ k',\ p';\ z)$ is continuous in the neighborhood of the singularity of the denominator. In addition to this term with a singularity of the type $\left(z+\varkappa_{23}^2-\frac{p_1^2}{2n_1}\right)^{-1}$, there appears a term with a singularity of the type $\left(z+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}\right)^{-1}$ and a term which contains the product of these two singularities. It is characteristic that (5.2) depends on k_{23} only via $\varphi_{23}(k_{23})$. Similarly, the term with a singularity of the type $\left(z+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}\right)^{-1}$ depends on k_{31}' only via $\overline{\varphi_{31}(k_{31}')}$, and the term with a singularity of the type $\left(z+\varkappa_{23}^2-\frac{p_1^2}{2n_1}\right)^{-1}\left(z+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}\right)^{-1}$ depends on k_{23} and k_{31}' only via $\varphi_{23}(k_{23})\overline{\varphi_{31}(k_{31}')}$. These will be referred to as the fundamental singularities.

In addition, there appears in (5.1) another singularity which is due to the vanishing of the expression $\frac{p_1^2}{2m_{31}}+\frac{(p_1,\ p_2')}{m_3}+\frac{p_2'^2}{2m_{23}}-z$. The position of this singularity depends on the magnitudes and directions of the variables p_1 and p_1'. We shall call such singularities secondary. They turn out to become weakened and generally disappear with increasing order of iteration.

In order to give a rigorous formulation we introduce now some necessary definitions. A kernel $\mathcal{Q}(k, p;\ k', p';\ z)$ is said to be of type $\mathfrak{Q}_{\alpha\beta}$ if it may be expressed in the form

$$\begin{aligned}\mathcal{Q}(k,\ p;\ k',\ p';\ z)=\mathcal{T}(k,\ p;\ k',\ p';\ z)+\\ +\mathcal{G}(k,\ p;\ p_\beta';\ z)\frac{\overline{\varphi_\beta(k_\beta')}}{z+\varkappa_\beta^2-\frac{p_\beta'^2}{2n_\beta}}+\frac{\varphi_\alpha(k_\alpha)}{z+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}}\tilde{\mathcal{G}}(p_\alpha;\ k',\ p';\ z)+\\ +\frac{\varphi_\alpha(k_\alpha)}{z+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}}\mathcal{H}(p_\alpha;\ p_\beta';\ z)\frac{\overline{\varphi_\beta(k_\beta')}}{z+\varkappa_\beta^2-\frac{p_\beta'^2}{2n_\beta}}.\end{aligned} \tag{5.3}$$

The kernels $\mathcal{T}(k, p;\ k', p';\ z)$, $\mathcal{G}(k, p;\ p_\beta';\ z)$, $\tilde{\mathcal{G}}(p_\alpha;\ k', p';\ z)$ and $\mathcal{H}(p_\alpha;\ p_\beta';\ z)$ are then called components of $\mathcal{Q}(k, p;\ k', p';\ z)$. A kernel $\mathcal{Q}(k, p;\ k', p';\ z)$ of type $\mathfrak{Q}_{\alpha\beta}$ is said to belong to the class $\mathfrak{Q}_{\alpha\beta}(\theta,\ \mu)$ if the kernel $\mathcal{T}(k, p;\ k', p';\ z)$ satisfies the following estimates

$$|\mathcal{T}(k,\ p;\ k',\ p';\ z)|\leqslant CN(k,\ p;\ \theta)\left(1+p_\beta'^2\right)^{-1}; \tag{5.4}$$

$$\begin{aligned}|\mathcal{T}(k+h,\ p+l;\ k'+h',\ p'+l';\ z+\Delta)-\mathcal{T}(k,\ p;\ k',\ p';\ z)|\leqslant\\ \leqslant CN(k,\ p;\ \theta)\left(1+p_\beta'^2\right)^{-1}[|h|^\mu+|l|^\mu+|h'|^\mu+|l'|^\mu+|\Delta|^\mu],\end{aligned} \tag{5.5}$$

and if the kernels $\mathcal{G}(k, p;\ p_\beta';\ z)$, $\tilde{\mathcal{G}}(p_\alpha;\ k', p';\ z)$ and $\mathcal{H}(p_\alpha;\ p_\beta';\ z)$ satisfy the estimates obtained from (5.4) and (5.5) on setting respectively $k_\beta'=0$, $k_\alpha=0$; and simultaneously $k_\alpha=0$ and $k_\beta'=0$. The estimating function $N(k, p;\ \theta)$ was defined in (3.23).

The kernels of the operators $\mathbf{Q}_{\gamma_1,\ldots,\gamma_{n+1}}^{(n)}(z)$ are studied in detail in § 6, where the following result is proved:

For $n \geqslant 4$ *the kernels* $\mathcal{Q}^{(n)}_{\alpha, \gamma_1, \ldots, \gamma_{n-1}, \beta}(k, p; k', p'; z)$ *of the operators* $\mathbf{Q}^{(n)}_{\alpha, \gamma_1, \ldots, \gamma_{n-1}, \beta}(z)$ *belong to the classes* $\mathfrak{Q}_{\alpha\beta}(\tilde{\theta}, \tilde{\mu})$ *with certain indices* $\tilde{\theta}$ *and* $\tilde{\mu}$, $\tilde{\theta} > \frac{1}{2}$, *uniformly over any finite region of the complex* z*-plane, denoted by* $\Pi_{-\varkappa}$, *to indicate that it is slit along the real axis from the point* $-\varkappa^2$, *where* $\varkappa^2 = \max(\varkappa^2_{23}, \varkappa^2_{31}, \varkappa^2_{12})$, *to* ∞. *When* $|z| \to \infty$, *the constants in the estimates of the type (5.4) and (5.5) cannot increase faster than some finite power of* $|z|$.

Let us now describe the function space in which we intend to study the system of equations (3.15). We consider several classes of functions. A function $\rho(k, p)$ is said to be of class $\mathfrak{M}(\theta, \mu)$ if it satisfies the conditions

$$|\rho(k, p)| \leqslant CN(k, p; \theta); \tag{5.6}$$

$$|\rho(k+h, p+l) - \rho(k, p)| \leqslant CN(k, p; \theta)(|h|^{\mu} + |l|^{\mu}). \tag{5.7}$$

A function $\sigma(p)$ is said to be of class $\mathfrak{N}(\theta, \mu)$ if it satisfies the conditions

$$|\sigma(p)| \leqslant C(1+|p|)^{-2(1+\theta)}; \tag{5.8}$$

$$|\sigma(p+l) - \sigma(p)| \leqslant C(1+|p|)^{-2(1+\theta)}|l|^{\mu}. \tag{5.9}$$

The function $(1+|p|)^{-2(1+\theta)}$ in (5. 8) and (5.9) is adjusted so that $(1+|p_\alpha|)^{2(1+\theta)} N(k, p; \theta)|_{k_\alpha=0}$ does not vanish and is uniformly bounded for any p_α, so that the estimating functions $(1+|p_\alpha|)^{-2(1+\theta)}$ and $N(k, p; \theta)|_{k_\alpha=0}$ are equivalent.

Consider the set of elements ω, consisting of the totality of "sextets" of functions

$$\omega = (\rho_{23}(k, p), \rho_{31}(k, p), \rho_{12}(k, p), \sigma_1(p_1), \sigma_2(p_2), \sigma_3(p_3)),$$

where $\rho_\alpha(k, p) \in \mathfrak{M}(\theta, \mu)$ and $\sigma_\alpha(p) \in \mathfrak{N}(\theta, \mu)$, $\alpha = 23, 31, 12$. This set will be a complete Banach space, which we designate $\mathfrak{B}(\theta, \mu)$, if the norm of ω is defined as

$$\|\omega\|_{\theta, \mu} = \sum_{\alpha} \sup\Big\{(1+|p_\alpha|)^{2(1+\theta)}\Big[|\sigma_\alpha(p_\alpha)| + \frac{|\sigma_\alpha(p_\alpha+l) - \sigma_\alpha(p_\alpha)|}{|l|^{\mu}}\Big] +$$
$$+ (N(k, p; \theta))^{-1}\Big[|\rho_\alpha(k, p)| + \frac{|\rho_\alpha(k+h, p) - \rho_\alpha(k, p)|}{|h|^{\mu}} +$$
$$+ \frac{|\rho_\alpha(k, p+l) - \rho_\alpha(k, p)|}{|l|^{\mu}}\Big]\Big\}.$$

The functions $\rho_\alpha(k, p)$ and $\sigma_\alpha(p_\alpha)$ will be called the components of the element ω. In the following we shall often use the function

$$\chi_\alpha(k, p; z) = \rho_\alpha(k, p) + \frac{\varphi_\alpha(k_\alpha)\sigma_\alpha(p_\alpha)}{z + \varkappa_\alpha^2 - \dfrac{p_\alpha^2}{2n_\alpha}} \tag{5.10}$$

formed from $\rho_\alpha(k, p)$ and $\sigma_\alpha(p_\alpha)$.

Consider the operator $\mathbf{A}(z)$, defined on the elements ω of $\mathfrak{B}(\theta, \mu)$ by

$$\omega' = \mathbf{A}(z)\omega \tag{5.11}$$

which means explicitly

$$\rho'_\alpha(k, p) = -\int \hat{t}_\alpha\left(k_\alpha, k'_\alpha, z - \frac{p_\alpha^2}{2n_\alpha}\right) \frac{\delta(p_\alpha - p'_\alpha)}{\dfrac{k'^2}{2m} + \dfrac{p'^2}{2n} - z} \times$$
$$\times \sum_{\beta \neq \alpha} \chi_\beta(k', p'; z)\, dk'dp'; \tag{5.12}$$

$$\sigma'_\alpha(p_\alpha) = -\int \overline{\varphi_\alpha(k'_\alpha)} \frac{\delta(p_\alpha - p'_\alpha)}{\dfrac{k'^2}{2m} + \dfrac{p'^2}{2n} - z} \sum_{\beta \neq \alpha} \chi_\beta(k', p'; z)\, dk'dp'. \tag{5.13}$$

Here $\rho'_\alpha(k, p)$ and $\sigma'_\alpha(p_\alpha)$ are the components of ω'.

By adding (5.13) multiplied by $\varphi_\alpha(k_\alpha)\left(z+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}\right)$, to (5.12) and denoting the function formed from $\rho'_\alpha(k, p)$ and $\sigma'_\alpha(p_\alpha)$ according to (5.10) by $\chi'_\alpha(k, p; z)$, we obtain

$$\chi'_\alpha(k, p; z) = -\int t_\alpha\left(k_\alpha, k'_\alpha, z-\frac{p_\alpha^2}{2n_\alpha}\right)\frac{\delta(p_\alpha-p'_\alpha)}{\frac{k'^2}{2m}+\frac{p'^2}{2n}-z}\times$$
$$\times\sum_{\beta\neq\alpha}\chi_\beta(k', p'; z)\,dk'dp'. \qquad (5.14)$$

If $\mathrm{Im}\, z\neq 0$ and $\theta>\frac{1}{2}$, then $\chi_\alpha(k, p; z)\in\mathfrak{H}$ and (5.14) may be written in the form

$$\chi_\alpha(z) = -\mathbf{T}_\alpha(z)\mathbf{R}_0(z)\sum_{\beta\neq\alpha}\chi_\beta(z). \qquad (5.15)$$

We observe that the operator $\mathbf{A}(z)$ is closely associated with the system of equations (3.15) which we will study. This somewhat crude derivation has been necessary for the step-by-step examination of the fundamental singularities of the kernels $\mathscr{W}_{\alpha\beta}(k, p; k', p'; z)$.

We reduce the study of the system (3.15) to the investigation of a second-kind equation in the space $\mathfrak{B}(\theta, \mu)$ with the operator $\mathbf{A}(z)$. We consider instead of the operators $\mathbf{W}_{\alpha\beta}(z)$ the operators $\tilde{\mathbf{W}}_{\alpha 3}(z)$, obtained from the former by subtracting the first three iterations of (3.15) (cf. (3.21)).

The system of equations for the operators $\tilde{\mathbf{W}}_{\alpha\beta}(z)$ has the form

$$\tilde{\mathbf{W}}_{\alpha\beta}(z) = \tilde{\mathbf{W}}^{(0)}_{\alpha\beta}(z) - \mathbf{T}_\alpha(z)\mathbf{R}_0(z)\sum_{\gamma\neq\alpha}\tilde{\mathbf{W}}_{\gamma\beta}(z), \qquad (5.16)$$

where

$$\tilde{\mathbf{W}}^{(0)}_{\alpha\beta}(z) = \mathbf{Q}^{(4)}_{\alpha\beta}(z) = \sum_{\gamma_1, \gamma_2, \gamma_3}{}' \mathbf{Q}^{(4)}_{\alpha, \gamma_1, \gamma_2, \gamma_3, \beta}(z). \qquad (5.17)$$

The kernels $\tilde{\mathscr{W}}^{(0)}_{\alpha\beta}(k, p; k', p'; z)$ belong, by the above proposition anticipating the analysis of § 6, to the classes $\mathfrak{Q}_{\alpha\beta}(\tilde{\theta}, \tilde{\mu})$. Their components are denoted by $\mathscr{F}^{(0)}_{\alpha\beta}(k, p; k', p'; z)$, $\mathscr{G}^{(0)}_{\alpha\beta}(k, p; p'_\beta; z)$, $\tilde{\mathscr{G}}^{(0)}_{\alpha\beta}(p_\alpha; k', p'; z)$ and $\mathscr{H}^{(0)}_{\alpha\beta}(p_\alpha; p'_\beta; z)$. For fixed β, k', p' and z, the totality of kernels $\mathscr{F}^{(0)}_{\alpha\beta}(k, p; k', p'; z)$ and $\tilde{\mathscr{G}}^{(0)}_{\alpha\beta}(p_\alpha; k', p'; z)$ may be regarded as an element in $\mathfrak{B}(\theta, \mu)$, $\theta\leqslant\tilde{\theta}$, $\mu\leqslant\tilde{\mu}$. We denote this element by $\omega^{(0)}_\beta(k', p'; z)$. We similarly assign the element $\omega^{(0)}_\beta(p'_\beta; z)$ to the totality of kernels $\mathscr{G}^{(0)}_{\alpha\beta}(k, p; p'_\beta; z)$ and $\mathscr{H}^{(0)}_{\alpha\beta}(p_\alpha; p'_\beta; z)$ with fixed β, p'_β and z. The estimates of the type (5.4) and (5.5) now become

$$\|\omega^{(0)}_\beta(k', p'; z)\|_{\theta_1, \mu_1}\leqslant C(|z|)(1+p_\beta'^2)^{-1};$$
$$\|\omega^{(0)}_\beta(k'+h', p'+l'; z+\Delta)-\omega^{(0)}_\beta(k', p'\ z)\|_{\theta_1, \mu_1}\leqslant$$
$$\leqslant C(|z|)(1+p_\beta'^2)^{-1}[|h'|^{\mu_2}+|l'|^{\mu_2}+|\Delta|^{\mu_2}];$$
$$\|\omega^{(0)}_\beta(p'_\beta, z)\|_{\theta_1, \mu_1}\leqslant C(|z|)(1+p_\beta'^2)^{-1};$$
$$\|\omega^{(0)}_\beta(p'_\beta+l'; z+\Delta)-\omega^{(0)}_\beta(p'_\beta; z)\|_{\theta_1, \mu_1}\leqslant$$
$$\leqslant C(|z|)(1+p_\beta'^2)^{-1}[|l'|^{\mu_2}+|\Delta|^{\mu_2}].$$

Here $\theta_1\leqslant\tilde{\theta}$, $\mu_1+\mu_2=\tilde{\mu}$.

Consider the following equations in $\mathfrak{B}(\theta, \mu)$, with $\theta<\tilde{\theta}$, $\mu<\tilde{\mu}$,

$$\omega_\beta(k', p'; z) = \omega^{(0)}_\beta(k', p'; z)+\mathbf{A}(z)\omega_\beta(k', p'; z), \qquad (5.18)$$
$$\omega_\beta(p'_\beta; z) = \omega^{(0)}_\beta(p'_\beta; z)+\mathbf{A}(z)\omega_\beta(p'_\beta; z) \qquad (5.19)$$

Let these equations be solved for some z by an $\omega_\beta(k', p'; z)$ and an $\omega_\beta(p'_\beta; z)$ in $\mathfrak{B}(\theta, \mu)$. We denote the components of these elements by $\mathscr{F}_{\alpha\beta}(k, p; k', p'; z)$,

$\tilde{\mathscr{G}}_{\alpha\beta}(p_\alpha; k', p'; z)$ and $\mathscr{G}_{\alpha\beta}(k, p; p'_\beta; z)$, $\mathscr{H}_{\alpha\beta}(p_\alpha; p'_\beta; z)$, and consider these kernels as the components of kernels $\mathscr{W}_{\alpha\beta}(k, p; k', p'; z)$ of the type $\mathfrak{Q}_{\alpha\beta}$. It is easily verified that these kernels satisfy the equations

$$\tilde{\mathscr{W}}_{\alpha\beta}(k, p; k', p'; z) = \tilde{\mathscr{W}}^{(0)}_{\alpha\beta}(k, p; k', p'; z) -$$
$$-\int t_\alpha\left(k_\alpha, k''_\alpha, z - \frac{p_\alpha^2}{2n_\alpha}\right) \frac{\delta(p_\alpha - p''_\alpha)}{\frac{k''^2}{2m} + \frac{p''^2}{2n} - z} \sum_{\gamma\neq\alpha} \tilde{\mathscr{W}}_{\gamma\beta}(k'', p''; k', p'; z)\, dk''dp''. \tag{5.20}$$

If $\operatorname{Im} z \neq 0$ and $\theta > \frac{1}{2}$, then, as was shown in § 3, the integral operators $\tilde{\mathbf{W}}_{\alpha\beta}(z)$ with these kernels are defined on $\mathfrak{D}$, and the system of equations (5.20) may be expressed in the form (5.16). This step reduces the investigation of the kernels $\tilde{\mathscr{W}}_{\alpha\beta}(k, p; k', p'; z)$ to examining the solvability of the equation

$$\omega = \omega^{(0)} - \mathbf{A}(z)\,\omega \tag{5.21}$$

in the space $\mathfrak{B}(\theta, \mu)$.

This problem is analyzed in § 7, and the main result may be stated as follows:

Let $\omega^{(0)} \in \mathfrak{B}(\theta', \mu')$, $\mu' < \bar{\mu}$, $\theta' < \bar{\theta}$. *Equation (5.21) has a unique solution in* $\mathfrak{B}(\theta, \mu)$ *for any* z *in the plane* $\Pi_{-\varkappa^2}$, *excluding a set of points* $z = \lambda_n$ *on the real axis which constitute the discrete spectrum of* $\mathbf{H}$. *We may take* θ *and* μ *as close as desired to* θ' *and* μ'. *The solution satisfies the estimate*

$$\|\omega\|_{\theta,\mu} \leqslant C(|z|)\,\|\omega^{(0)}\|_{\theta',\mu'}, \tag{5.22}$$

uniformly in z *over any finite domain containing none of the singular points* λ_n. *The set* $\{\lambda_n\}$ *is denumerable, closed, and bounded, and may only have limit points at* $-\varkappa_\alpha^2$, $\alpha = 23, 31, 12$.

The statements made so far in this section (to be proved in §§ 6, 7) may be summed up in the following theorem:

Theorem 5.1. *Let the conditions* A_{θ_0}, B_{μ_0}, R *and* C *be fulfilled for all the three potentials* $v_\alpha(k)$, $\alpha = 23, 31, 12$, *with* $\theta_0 > \frac{1}{2}$ *and* $\mu_0 > \frac{1}{2}$. *Then the* $\tilde{\mathbf{W}}_{\alpha\beta}(z)$ *defined by (3.21), (3.22), (3.17) are integral operators whose kernels* $\tilde{\mathscr{W}}_{\alpha\beta}(k, p; k', p'; z)$ *belong to the classes* $\mathfrak{Q}_{\alpha\beta}(\theta, \mu)$ *with certain indices* θ *and* μ, *whereby we may choose* $\theta > \frac{1}{2}$. *Estimates of the type (5.4) and (5.5) hold for these kernels uniformly in* z *over any finite region of the complex plane* $\Pi_{-\varkappa^2}$, *slit along the positive real axis from the point* $-\varkappa^2$ *to* ∞, *provided that this region contains none of the singular points* λ_n *— the discrete spectrum of* $\mathbf{H}$ *— nor certain neighborhoods of them. The set of singular points* λ_n *lies in a finite interval, is denumerable, closed and may have as limit points only the points* $-\varkappa_\alpha^2$, $\alpha = 23, 31, 12$.

Along with the kernels $\mathscr{F}_{\alpha\beta}$, $\mathscr{G}_{\alpha\beta}$, $\tilde{\mathscr{G}}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$ (here written for brevity without their arguments) which serve to construct $\tilde{\mathscr{W}}_{\alpha\beta}$ in the manner of (5.3), we shall also need the kernels

$$\mathscr{K}_{\alpha\beta}(k, p: p'_\beta; z) = \mathscr{G}_{\alpha\beta}(k, p; p'_\beta; z) + \frac{\varphi_\alpha(k_\alpha)}{z + \varkappa_\alpha^2 - \frac{p_\alpha^2}{2n_\alpha}} \mathscr{H}_{\alpha\beta}(p_\alpha; p'_\beta; z); \tag{5.23}$$

$$\tilde{\mathscr{K}}_{\alpha\beta}(p_\alpha; k', p'; z) = \tilde{\mathscr{G}}_{\alpha\beta}(p_\alpha; k', p'; z) + \mathscr{H}_{\alpha\beta}(p_\alpha; p'_\beta; z) \frac{\overline{\varphi_\beta(k'_\beta)}}{z + \varkappa_\beta^2 - \frac{p'^2_\beta}{2n_\beta}}. \tag{5.24}$$

We denote by $\mathscr{K}^{(0)}_{\alpha\beta}$ and $\tilde{\mathscr{K}}^{(0)}_{\alpha\beta}$ the kernels formed in a similar fashion from the pairs $\mathscr{G}^{(0)}_{\alpha\beta}, \mathscr{H}^{(0)}_{\alpha\beta}$ and $\tilde{\mathscr{G}}^{(0)}_{\alpha\beta}, \tilde{\mathscr{H}}^{(0)}_{\alpha\beta}$, consisting of components of the $\tilde{\mathscr{W}}'^{(0)}_{\alpha\beta}$. It is obvious from relations similar to (5.3) for the $\tilde{\mathscr{W}}_{\alpha\beta}$ that in order to define the $\mathscr{K}_{\alpha\beta}$ or $\tilde{\mathscr{K}}_{\alpha\beta}$ it is necessary to separate in $\tilde{\mathscr{W}}_{\alpha\beta}$ the terms which are singular when $z+\varkappa_\beta^2-\frac{p_\beta'^2}{2n_\beta}\to 0$ or $z+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}\to 0$, respectively. It turns out that the kernels $\mathscr{K}_{\alpha\beta}$ and $\tilde{\mathscr{K}}_{\alpha\beta}$ may also be formed from $\tilde{\mathscr{W}}_{\alpha\beta}$ in a different way, indicated in the following lemma.

Lemma 5.1. *The following relations hold for any z with* $\operatorname{Im} z \neq 0$

$$\int \tilde{\mathscr{W}}_{\alpha\beta}(k,\ p;\ k',\ p';\ z)\frac{\psi_\beta(k_\beta')}{\frac{k'^2}{2m}+\frac{p'^2}{2n}-z}\,dk_\beta' = \mathscr{K}_{\alpha\beta}(k,\ p;\ p_\beta';\ z)\frac{1}{z+\varkappa_\beta^2-\frac{p_\beta'^2}{2n_\beta}}; \tag{5.25}$$

$$\int \frac{\overline{\psi_\alpha(k_\alpha)}}{\frac{k^2}{2m}+\frac{p^2}{2n}-z}\tilde{\mathscr{W}}_{\alpha\beta}(k,\ p;\ k',\ p';\ z)\,dk_\alpha = \frac{1}{z+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}}\tilde{\mathscr{K}}_{\alpha\beta}(p_\alpha;\ k',\ p';\ z). \tag{5.26}$$

Proof. We first prove that

$$\int t_\alpha(k,\ k',\ z)\left(\frac{k'^2}{2m}-z\right)^{-1}\psi(k')\,dk' = \frac{\varphi_\alpha(k)}{z+\varkappa_\alpha^2};\ \alpha=23,\ 31,\ 12. \tag{5.27}$$

(We omit the index α for convenience.) The function $\psi(k)$, we recall, is an eigenfunction of $\mathbf{h}$ whose eigenvalue is $-\varkappa^2$, so that if $\mathbf{r}(z)$ is the resolvent of $\mathbf{h}$, then

$$\mathbf{r}(z)\psi = \frac{1}{-\varkappa^2-z}\psi.$$

Let us write this relation explicitly, using the expression (2.10) for the resolvent $\mathbf{r}(z)$ in terms of the operator $\mathbf{t}(z)$ with the kernel $t(k,\ k',\ z)$. We obtain

$$\left(\frac{k^2}{2m}-z\right)^{-1}\psi(k)+\left(\frac{k^2}{2m}-z\right)^{-1}\int t(k,\ k',\ z)\left(\frac{k'^2}{2m}-z\right)^{-1}\psi(k')\,dk' = -\frac{\psi(k)}{z+\varkappa^2}. \tag{5.28}$$

Multiplying this by $\frac{k^2}{2m}-z$ and collecting terms, we obtain

$$\int t(k,\ k',\ z)\left(\frac{k'^2}{2m}-z\right)^{-1}\psi(k')\,dk' = \left(\frac{k^2}{2m}+\varkappa^2\right)\frac{\psi(k)}{z+\varkappa^2},$$

which is exactly (5.27), since $\varphi(k)=\left(\frac{k^2}{2m}+\varkappa^2\right)\psi(k)$.

We now replace in (5.27) z by z and take the complex conjugate of both sides. After simple changes of the symbols of the variables and using the symmetry relation (2.15) for $t(k,\ k',\ z)$, we obtain

$$\int \overline{\psi_\alpha(k)}\left(\frac{k^2}{2m}-z\right)^{-1}t_\alpha(k,\ k',\ z)\,dk = \frac{\overline{\varphi_\alpha(k)}}{z+\varkappa^2}. \tag{5.29}$$

Let us show that the $\tilde{\mathscr{W}}^{(0)}_{\alpha\beta}$, $\mathscr{K}^{(0)}_{\alpha\beta}$ and $\tilde{\mathscr{K}}^{(0)}_{\alpha\beta}$ satisfy relations of the type (5.23) and (5.24). We note that the kernels $\tilde{\mathscr{W}}^{(0)}_{\alpha\beta}$ may be represented by an integral of the form

$$\int t_\alpha\left(k_\alpha,\ k_\alpha^{(1)},\ z-\frac{p_\alpha^2}{2n_\alpha}\right)\dots t_\beta\left(k_\beta^{(4)},\ k_\beta',\ z-\frac{p_\beta'^2}{2n_\beta}\right)dk^{(1)}dp^{(1)}\dots dk^{(4)}dp^{(4)}. \tag{5.30}$$

We have written explicitly those factors of the integrand which determine the dependence on k_α and k'_β. The kernel $\mathcal{K}^{(0)}_{\alpha\beta}$ may be represented by a similar integral in which the kernel $t_\beta\left(k^{(4)}_\beta, k'_\beta, z-\frac{p'^2_\beta}{2n_\beta}\right)$ is replaced by the function $\varphi_\beta(k^{(4}_\beta)$.

We multiply (5.30) by $\left(\frac{k'^2}{2m}+\frac{p'^2}{2n}-z\right)^{-1}\psi_\beta(k'_\beta)$ and integrate with respect to k'_β. All the integrals converge absolutely when $\operatorname{Im} z \neq 0$, so we may change the order of integration and integrate first with respect to k'_β. By (5.27), the kernel $t_\beta\left(k^{(4)}_\beta, k'_\beta, z-\frac{p'^2_\beta}{2n\beta}\right)$ is then replaced by the function $\varphi_\beta(k^{(4)}_\beta)\left(z+\varkappa^2_\beta-\frac{p'^2_\beta}{2n_\beta}\right)^{-1}$, which leads to a relation of the type (5.25) for the $\tilde{\mathcal{W}}^{(0)}_{\alpha\beta}$ and $\mathcal{K}^{(0)}_{\alpha\beta}$. One can similarly derive a relation of the type (5.26) for the $\tilde{\mathcal{W}}^{(0)}_{\alpha\beta}$ and $\tilde{\mathcal{K}}^{(0)}_{\alpha\beta}$, by multiplying (5.30) by $\psi_\alpha(k_\alpha)\left(\frac{k^2}{2m}+\frac{p^2}{2n}-z\right)^{-1}$ integrating with respect to k_α and applying (5.29).

We must still prove the relations (5.25) and (5.26) themselves. Multiplying (5.20) by $\left(\frac{k'^2}{2m}+\frac{p'^2}{2n}-z\right)^{-1}\psi_\beta(k'_\beta)$ and integrating with respect to k'_β, we find that the kernels

$$\mathcal{N}_{\alpha\beta}(k, p; p'_\beta; z)=\int\tilde{\mathcal{W}}_{\alpha\beta}(k, p; k', p'; z)\left(\frac{k'^2}{2m}+\frac{p'^2}{2n}-z\right)^{-1}\psi_\beta(k'_\beta)\,dk'_\beta$$

satisfy the equations

$$\mathcal{N}_{\alpha\beta}(k, p; p'_\beta; z)=\mathcal{K}^{(0)}_{\alpha\beta}(k, p; p'_\beta; z)\frac{1}{z+\varkappa^2_\beta-\frac{p'^2_\beta}{2n_\beta}}-$$
$$-\int t_\alpha\left(k_\alpha, k''_\alpha, z-\frac{p^2_\alpha}{2n_\alpha}\right)\frac{\delta(p_\alpha-p''_\alpha)}{\frac{k''^2}{2m}+\frac{p''^2}{2n}-z}\sum_{\gamma\neq\alpha}\mathcal{N}_{\gamma\beta}(k'', p''; p'_\beta; z)\,dk''dp''. \qquad (5.31)$$

It follows from the properties of the $\tilde{\mathcal{W}}_{\alpha\beta}$ that the $\mathcal{N}_{\alpha\beta}$ may be represented as

$$\mathcal{N}_{\alpha\beta}(k, p; p'_\beta; z)=\rho_{\alpha\beta}(k, p; p'_\beta; z)+\frac{\varphi_\alpha(k_\alpha)\,\sigma_{\alpha\beta}(p_\alpha; p'_\beta; z)}{z+\varkappa^2_\alpha-\frac{p^2_\alpha}{2n_\alpha}},$$

where $\rho_{\alpha\beta}\in\mathfrak{M}(\theta, \mu)$ and $\sigma_{\alpha\beta}\in\mathfrak{N}(\theta, \mu)$ for fixed β, p'_β and z. By definition of the kernels $\mathcal{K}_{\alpha\beta}$ and equation (5.19), it follows that the system (5.31) is also satisfied by the kernels $\mathcal{K}_{\alpha\beta}(k, p; p'_\beta; z)\left(z+\varkappa^2_\beta-\frac{p'^2_\beta}{2n_\beta}\right)^{-1}$. Relation (5.25) then follows by virtue of the uniqueness of solution of this system within the considered class of kernels.

To prove (5.26), note that

$$\tilde{\mathcal{K}}_{\alpha\beta}(p_\alpha; k', p'; z)=\tilde{\mathcal{K}}^{(0)}_{\alpha\beta}(p_\alpha; k', p'; z)-$$
$$-\int\overline{\varphi_\alpha(k''_\alpha)}\frac{\delta(p_\alpha-p''_\alpha)}{\frac{k''^2}{2m}+\frac{p''^2}{2n}-z}\sum_{\gamma\neq\alpha}\tilde{\mathcal{W}}_{\gamma\beta}(k'', p''; k', p'; z)\,dk''dp'', \qquad (5.32)$$

by definition of $\mathcal{H}_{\alpha\beta}$, $\mathcal{J}_{\alpha\beta}$ and $\tilde{\mathcal{K}}_{\alpha\beta}$. We multiply (5.20) by $\psi_\alpha(k_\alpha)\left(\frac{k^2}{2m}+\frac{p^2}{2n}-z\right)^{-1}$, integrate with respect to k_α, and in view of (5.29) and a relation of the type

(5.26) for the kernels $\widetilde{\mathscr{W}}^{(0)}_{\alpha\beta}$ and $\mathscr{K}^{(0)}_{\alpha\beta}$, find that the kernels

$$\mathscr{N}_{\alpha\beta}(p_\alpha;\ k',\ p';\ z)=\int\frac{\overline{\psi_\alpha(k_\alpha)}}{\frac{k^2}{2m}+\frac{p^2}{2n}-z}\widetilde{\mathscr{W}}_{\alpha\beta}(k,\ p;\ k',\ p';\ z)\,dk_\alpha \tag{5.33}$$

satisfy the relations

$$\mathscr{N}_{\beta\beta}(p_\alpha;\ k',\ p';\ z)=\frac{1}{z+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}}\mathscr{K}^{(0)}_{\alpha\beta}(p_\alpha;\ k',\ p';\ z)-$$

$$-\int\overline{\varphi_\alpha(k''_\alpha)}\frac{\delta(p_\alpha-p''_\alpha)}{\frac{k''^2}{2m}+\frac{p''^2}{2n}-z}\sum_{\gamma\neq\alpha}\widetilde{\mathscr{W}}_{\alpha\beta}(k'',\ p'';\ k',\ p';\ z)\,dk''dp''. \tag{5.34}$$

Comparing (5.32), (5.33) and (5.34), we obtain (5.26).

This completes the proof.

Lemma 5.2. The following relations are valid

$$\mathscr{K}_{\alpha\beta}(k,\ p;\ p'_\beta;\ z)=\overline{\mathscr{K}_{\beta\alpha}(p'_\beta;\ k,\ p;\ \bar{z})}. \tag{5.35}$$

Proof. We make use of the symmetry relations (3.32) for the kernels $\mathscr{W}_{\alpha\beta}$. It is easily verified that similar relations hold for the $\widetilde{\mathscr{W}}_{\alpha\beta}$, too, viz.,

$$\widetilde{\mathscr{W}}_{\alpha\beta}(k,\ p;\ k',\ p';\ z)=\overline{\widetilde{\mathscr{W}}_{\beta\alpha}(k',\ p';\ k,\ p;\ \bar{z})}. \tag{5.36}$$

We replace in (5.26) z by $\bar{z}$, interchange α and β, and also interchange the primed and unprimed variables. Taking the complex conjugate of the new expression, we get

$$\int\overline{\widetilde{\mathscr{W}}_{\beta\alpha}(k',\ p';\ k,\ p;\ \bar{z})}\frac{\psi_\beta(k'_\beta)}{\frac{k'^2}{2m}+\frac{p'^2}{2n}-z}dk'_\beta=\overline{\mathscr{K}_{\beta\alpha}(p'_\beta;\ k,\ p;\ \bar{z})}\frac{1}{z+\varkappa_\beta^2-\frac{p'^2_\beta}{2n_\beta}}. \tag{5.37}$$

Comparing (5.37) with (5.25) and keeping in mind (5.36), we infer (5.35), which completes the proof.

The relations (5.25), (5.26) and (5.35) will be applied in § 9. To conclude this section we now show that the assertion made in § 3 concerning the behavior of the kernels $\widetilde{\mathscr{W}}_{\alpha\beta}(k,\ p;\ k', p';\ z)$ for $\operatorname{Im} z\neq 0$, follows from Theorem 5.1. To this end we have to show that the estimates (5.4) and (5.5) are valid for the complete kernel $\widetilde{\mathscr{W}}_{\alpha\beta}(k,\ p;\ k',\ p';\ z)$ and not only for its component $\mathscr{T}_{\alpha\beta}(k,\ p;\ k',\ p';\ z)$. Consider, say, the kernel

$$\frac{\varphi_{23}(k_{23})}{z+\varkappa_{23}^2-\frac{p_1^2}{2n_1}}\mathscr{G}_{23,\,31}(p_1;\ k',\ p';\ z). \tag{5.38}$$

For $\operatorname{Im} z\neq 0$

$$\left|\left(z+\varkappa_{23}^2-\frac{p_1^2}{2n_1}\right)^{-1}\right|\leqslant\frac{1}{|\operatorname{Im} z|}$$

and

$$\varphi_{23}(k_{23})\mathscr{G}_{23,\,31}(p_1;\ k',\ p';\ z)|\leqslant C(1+|k_{23}|)^{-(1+\theta)}(1+|p_1|)^{-2(1+\theta)}\times$$

$$\times\left(1+p_2'^2\right)^{-1}\leqslant C(1+|p_1|)^{-(1+\theta)}(1+|p_2|)^{-(1+\theta)}\left(1+p_2'^2\right)^{-1}\leqslant$$

$$\leqslant CN(k,\ p;\ \theta)\left(1+p_2'^2\right)^{-1}.$$

We have used here the estimate

$$|\mathscr{G}_{23,\,31}(p_1;\ k',\ p';\ z)|\leqslant C(1+|p_1|)^{-2}(^{1+\theta})(1+p_2'^2)^{-1},$$

which is valid for $\mathscr{G}_{23,\,31}$ for any z with $\operatorname{Im} z\neq 0$, since the singular points are restricted to the real axis. Thus (5.38) satisfies an estimate of the type

(5.4). The remaining estimates are similarly derived, so that the assertion made in § 3 may be considered to be proved.

§ 6. Estimates of the kernels $\mathcal{Q}^{(n)}_{\gamma_1,\ldots,\gamma_{n+1}}(k, p; k', p'; z)$

In this section we shall study in detail the behavior of the kernels $\mathcal{Q}^{(n)}_{\gamma_1,\ldots,\gamma_{n+1}}(k, p; k', p'; z)$ of the operators $\mathbf{Q}^{(n)}_{\gamma_1,\ldots,\gamma_{n+1}}(z)$, introduced in § 3. Let $\gamma_1, \ldots, \gamma_{n+1}$ be a sequence of $n+1$ indices, each of which runs over the values 23, 31, 12, while $\gamma_i \neq \gamma_{i+1}$, $i=1, \ldots, n$. The operators $\mathbf{Q}^{(n)}_{\gamma_1,\ldots,\gamma_{n+1}}(z)$ are defined for $\mathbf{Im}\, z \neq 0$ by

$$\mathbf{Q}^{(n)}_{\gamma_1,\ldots,\gamma_{n+1}}(z)=(-1)^n \mathbf{T}_{\gamma_1}(z)\mathbf{R}_0(z)\mathbf{T}_{\gamma_2}(z)\ldots \mathbf{T}_{\gamma_n}(z)\mathbf{R}_0(z)\mathbf{T}_{\gamma_{n+1}}(z). \tag{6.1}$$

These are integral operators, whose kernels we will denote by $\mathcal{Q}^{(n)}_{\gamma_1,\ldots,\gamma_{n+1}}(k, p; k', p'; z)$. By the definition (6.1), these kernels are represented by the integrals

$$\mathcal{Q}^{(n)}_{\gamma_1,\ldots,\gamma_{n+1}}(k, p; k', p'; z)=(-1)^n \int t_{\gamma_1}\left(k_{\gamma_1}, k^{(1)}_{\gamma_1}, z-\frac{p^2_{\gamma_1}}{2n_{\gamma_1}}\right)\frac{\delta\left(p_{\gamma_1}-p^{(1)}_{\gamma_1}\right)}{\frac{k^{(1)^2}}{2m}+\frac{p^{(1)^2}}{2n}-z}\ldots$$

$$\ldots\frac{\delta\left(p^{(n)}_{\gamma_{n+1}}-p'_{\gamma_{n+1}}\right)}{\frac{k^{(n)^2}}{2m}+\frac{p^{(n)^2}}{2n}-z}\, t_{\gamma_{n+1}}\left(k^{(n)}_{\gamma_{n+1}}, k'_{\gamma_{n+1}}, z-\frac{p'^2_{\gamma_{n+1}}}{2n_{\gamma_{n+1}}}\right) dk^{(1)}dp^{(1)}\ldots dk^{(n)}dp^{(n)}. \tag{6.2}$$

In order to investigate these integrals we must first derive some estimates of the kernels $t_\alpha\left(k_\alpha, k'_\alpha, z-\frac{p^2_\alpha}{2n_\alpha}\right)$.

Lemma 6.1. The kernels $t_\alpha\left(k_\alpha, k'_\alpha, z-\frac{p^2_\alpha}{2n_\alpha}\right)$ *may be represented in the form*

$$t_\alpha\left(k_\alpha, k'_\alpha, z-\frac{p^2_\alpha}{2n_\alpha}\right)=\frac{\varphi_\alpha(k_\alpha)\overline{\varphi_\alpha(k'_\alpha)}}{z+\varkappa^2_\alpha-\frac{p^2_\alpha}{2n_\alpha}}+u_\alpha(k_\alpha, k'_\alpha, p_\alpha; z), \tag{6.3}$$

where $\varphi_\alpha(k)\in\mathfrak{m}(\theta_0, \mu_0)$ *and* $u_\alpha(k_\alpha, k'_\alpha, p_\alpha; z)$ *satisfies the estimates*

$$|u_\alpha(k_\alpha, k'_\alpha, p_\alpha; z)|\leqslant C(1+|k_\alpha-k'_\alpha|)^{-(1+\theta'_0)}; \tag{6.4}$$

$$|u_\alpha(k_\alpha+h, k'_\alpha+h', p_\alpha; z+\Delta)-u_\alpha(k_\alpha, k'_\alpha, p_\alpha; z)|\leqslant$$

$$\leqslant C(1+|k_\alpha-k'_\alpha|)^{-(1+\theta'_0)}\left[|h|^{\mu'_0}+|h'|^{\mu'_0}+|\Delta|^{\nu}\right]; \tag{6.5}$$

$$|u_\alpha(k_\alpha, k'_\alpha, p_\alpha+l; z)-u_\alpha(k_\alpha, k'_\alpha, p_\alpha; z)|\leqslant$$

$$\leqslant C(1+|z|)^{\frac{\nu}{2}}(1+|k_\alpha-k'_\alpha|)^{-(1+\theta'_0)}|l|^{\nu}, \tag{6.6}$$

and θ'_0, μ'_0 *and* ν *are less than, but as close as desired to* θ_0, μ_0 *and* $\frac{1}{2}$, *respectively.*

Proof. We use the representation (4.50) for the kernel $t_\alpha(k_\alpha, k'_\alpha, z)$ and write

$$u_\alpha(k_\alpha, k'_\alpha, p_\alpha; z)=\hat{t}_\alpha\left(k_\alpha, k'_\alpha, z-\frac{p^2_\alpha}{2n_\alpha}\right). \tag{6.7}$$

All the assertions of the lemma, except the estimate (6.6), now follow directly from Theorem 4.3. To prove (6.6) we consider the two cases:

1. $p_\alpha^2 \leqslant 4M|z|+1$, where $M=\max(m_1, m_2, m_3)$. Then we obtain with the help of (4.52)

$$\begin{aligned} &|u_\alpha(k_\alpha, k'_\alpha, p_\alpha+l;\ z)-u_\alpha(k_\alpha, k'_\alpha, p_\alpha;\ z)| \leqslant \\ &\leqslant C(1+|k_\alpha-k'_\alpha|)^{-(1+\theta'_0)}|(p_\alpha+l)^2-p_\alpha^2|^\nu \leqslant \\ &\leqslant C(1+|k_\alpha-k'_\alpha|)^{-(1+\theta'_0)}|p_\alpha|^\nu|l|^\nu \leqslant \\ &\leqslant C(1+|z|)^{\frac{\nu}{2}}(1+|k_\alpha-k'_\alpha|)^{-(1+\theta'_0)}|l|^\nu. \end{aligned}$$

2. $p_\alpha^2 \geqslant 4M|z|+1$. Then

$$\left|z-\frac{p_\alpha^2}{2n_\alpha}\right| \geqslant \frac{1}{2M}p_\alpha^2-|z| \geqslant \frac{1}{4M}(p_\alpha^2+1),$$

and hence, using (4.53),

$$\begin{aligned} &|u_\alpha(k_\alpha, k'_\alpha, p_\alpha+l;\ z)-u_\alpha(k_\alpha, k'_\alpha, p_\alpha;\ z)| \leqslant \\ &\leqslant C(1+|k_\alpha-k'_\alpha|)^{-(1+\theta'_0)}|p_\alpha||l|\left(1+\left|z-\frac{p_\alpha^2}{2n_\alpha}\right|\right)^{-1} \leqslant \\ &\leqslant C(1+|k_\alpha-k'_\alpha|)^{-(1+\theta'_0)}|l|. \end{aligned}$$

This completes the proof.

In all the estimates of this and the next section we use only those properties ot the kernels $t_\alpha\left(k_\alpha, k'_\alpha, z-\frac{p_\alpha^2}{2n_\alpha}\right)$ which are summed up in the last lemma. The primes on μ_0 and θ_0 will be omitted.

Let us now examine the behavior of the kernels $\mathcal{Q}^{(n)}_{\gamma_1,\ldots,\gamma_{n+1}}(k, p; k', p'; z)$. The δ-functions account for all the integrations in (6.2) when $n=1$, so that the kernels $\mathcal{Q}^{(1)}_{\alpha\beta}(k, p;\ k', p';\ z)$ may be expressed explicitly by means of the $t_\alpha\left(k_\alpha, k'_\alpha, z-\frac{p_\alpha^2}{2n_\alpha}\right)$. Such an expression for $\mathcal{Q}^{(1)}_{23,31}$ was given before in § 5 (cf. (5.1)). Here we start with the kernels $\mathcal{Q}^{(2)}_{\alpha,\beta,\gamma}$.

We take for example $\alpha=23, \beta=31, \gamma=12$ and, in order to avoid encumbering the formulas, we omit the indices and denote temporarily the operator $\mathbf{Q}^{(2)}_{23,31,12}(z)$ by $\mathbf{Q}(z)$ and its kernel by $\mathcal{Q}(k, p;\ k', p';\ z)$. Taking the integration variables in (6.2), with $n=2$ and $\gamma_1=23$, $\gamma_2=31$, $\gamma_3=12$, as $p_1^{(1)}$, $p_2^{(1)}=q$, $p_2^{(2)}$, $p_3^{(2)}$, we obtain the following expression for $\mathcal{Q}(k, p;\ k', p';\ z)$

$$\begin{aligned} \mathcal{Q}(k, p;\ k', p';\ z)=\int t_{23}\left(k_{23}, -q-\frac{m_2}{m_2+m_3}p_1, z-\frac{p_1^2}{2n_1}\right) \times \\ \times\left[\frac{1}{2m_{23}}\left(q+\frac{m_2}{m_2+m_3}p_1\right)^2+\frac{p_1^2}{2n_1}-z\right]^{-1} \times \\ \times t_{31}\left(p_1+\frac{m_1}{m_3+m_1}q, -p'_3-\frac{m_3}{m_3+m_1}q, z-\frac{q^2}{2n_2}\right) \times \\ \times\left[\frac{1}{2m_{12}}\left(q+\frac{m_2}{m_1+m_2}p'_3\right)^2+\frac{p_3'^2}{2n_3}-z\right]^{-1} \times \\ \times t_{12}\left(q+\frac{m_2}{m_1+m_2}p'_3, k'_{12}, z-\frac{p_3'^2}{2n_3}\right)dq. \end{aligned}$$

Substituting here the expression (6.3) for t_{23} and t_{12}, we deduce that the kernel $\mathcal{Q}(k, p;\ k', p';\ z)$ is of the type $\mathfrak{Q}_{23,12}$. We denote its components by $\mathcal{T}$, $\mathcal{G}$, $\tilde{\mathcal{G}}$

and $\mathcal{H}$. Now $\mathcal{T}(k, p; k', p'; z)$ has the representation

$$\mathcal{T}(k, p; k', p'; z) = \int \hat{t}_{23}\left(k_{23}, -q - \frac{m_2}{m_2+m_3}p_1, z - \frac{p_1^2}{2n_1}\right) \times$$

$$\times \left[\frac{1}{2m_{23}}\left(q + \frac{m_2}{m_2+m_3}p_1\right)^2 + \frac{p_1^2}{2n_1} - z\right]^{-1} \times$$

$$\times \left[\hat{t}_{31}\left(p_1 + \frac{m_1}{m_3+m_1}q, -p_3' - \frac{m_3}{m_3+m_1}q, z - \frac{q^2}{2n_2}\right) + \right.$$

$$\left. + \frac{\varphi_{31}\left(p_1 + \frac{m_1}{m_3+m_1}q\right)\overline{\varphi_{31}\left(-p_3' - \frac{m_3}{m_3+m_1}q\right)}}{z + \varkappa_{31}^2 - \frac{q^2}{2n_2}}\right] \times$$

$$\times \left[\frac{1}{2m_{12}}\left(q + \frac{m_2}{m_1+m_2}p_3'\right)^2 + \frac{p_3'^2}{2n_3} - z\right]^{-1} \times$$

$$\times \hat{t}_{12}\left[q + \frac{m_2}{m_1+m_2}p_3', k_{12}', z - \frac{p_3'^2}{2n_3}\right) dq. \tag{6.8}$$

The corresponding expressions for $\mathcal{G}$, $\tilde{\mathcal{G}}$, and $\mathcal{H}$ are obtained by replacing here respectively $\hat{t}_{12}$ by $\varphi_{12}\left(q + \frac{m_2}{m_1+m_2}p_3'\right)$; $\hat{t}_{23}$ by $\overline{\varphi_{23}\left(-q - \frac{m_2}{m_2+m_3}p_1\right)}$; and doing both simultaneously.

The integral (6.8) consists of two terms, one of which contains in the integrand a product of two, and the other of three generally singular denominators. Some of these factors are singular in different parts of the domain of integration. We have, for example,

$$\left[\frac{1}{2m_{23}}\left(q + \frac{m_2}{m_2+m_3}p_1\right)^2 + \frac{p_1^2}{2n_1} - z\right] + \left[z + \varkappa_{31}^2 - \frac{q^2}{2n_2}\right] =$$

$$= \left[\frac{1}{2m_{31}}\left(p_1 + \frac{m_1}{m_3+m_1}q\right)^2 + \frac{q^2}{2n_2} - z\right] + \left[z + \varkappa_{31}^2 - \frac{q^2}{2n^2}\right] =$$

$$= \frac{1}{2m_{31}}\left(p_1 + \frac{m_1}{m_3+m_1}q\right)^2 + \varkappa_{31}^2 \geqslant \varkappa_{31}^2. \tag{6.9}$$

We have applied here the formulas which relate the k-type and p-type variables (cf. § 1). Similarly

$$\left[\frac{1}{2m_{12}}\left(q + \frac{m_2}{m_1+m_2}p_3'\right)^2 + \frac{p_3'^2}{2n_3} - z\right] + \left[z + \varkappa_{31}^2 - \frac{q^2}{2n_2}\right] =$$

$$= \frac{1}{2m_{31}}\left(p_3' + \frac{m_3}{m_3+m_1}q\right)^2 + \varkappa_{31}^2 \geqslant \varkappa_{31}^2. \tag{6.10}$$

We observe that the left-hand sides of (6.9) and (6.10) cannot vanish simultaneously.

If the variables p_1, p_3' or q are sufficiently large, then the denominators in (6.8) are not singular. Thus let, for example, $q^2 \geqslant 4M|z| + 1$. We then have

$$\left|\frac{1}{2m_{23}}\left(q + \frac{m_2}{m_2+m_3}p_1\right)^2 + \frac{p_1^2}{2n_1} - z\right| = \left|\frac{1}{2}\left(\frac{1}{m_3} + \frac{1}{m_1}\right)p_1^2 + \right.$$

$$\left. + \frac{(p_1, q)}{m_3} + \frac{1}{2}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)q^2 - z\right| \geqslant \frac{p_1^2}{4m_1} + \frac{q^2}{4m_2} +$$

$$+ \left(\frac{q^2}{4m_2} + \frac{p_1^2}{4m_1} - |z|\right) \geqslant \frac{1}{4M}(p_1^2 + q^2 + 1). \tag{6.11}$$

It is evident from the derivation of this estimate that it is also valid when $p_1^2 \geqslant 4M|z|+1$. It may be deduced similarly that if $q^2 \geqslant 4M|z|+1$ or $p_3'^2 \geqslant 4M|z|+1$, then

$$\left|\frac{1}{2m_{12}}\left(q+\frac{m_2}{m_1+m_2}p_3'\right)^2+\frac{p_3'^2}{2n_3}-z\right| \geqslant \frac{1}{4M}\left(q^2+p_3'^2+1\right). \tag{6.12}$$

Finally, if $q^2 \geqslant 4M(|z|+\varkappa^2)+1$, where $\varkappa^2=\max(\varkappa_\alpha^2)$, then

$$\left|z+\varkappa_{31}^2-\frac{q^2}{2n_2}\right| \geqslant \frac{1}{4M}(1+q^2). \tag{6.13}$$

Let us now examine the properties of the numerator in the integrand of (6.8). All the functions appearing here are Hölder functions with an index not less than the ν of Lemma 6.1. The following may serve as an estimating function for the numerators of both terms

$$\begin{aligned}&(1+|z|)^{\frac{3\nu}{2}}(1+|p_1-p_1'|)^{-(1+\theta_0)}(1+|p_2-q|)^{-(1+\theta_0)}\times\\ &\times[(1+|p_1+q+p_3'|)^{-(1+\theta_0)}+(1+|q-p_2'|)^{-(1+\theta_0)}].\end{aligned} \tag{6.14}$$

This is obvious for the first term, since the product of the estimating functions for each of the factors in the numerator

$$\left[(1+|z|)^{\frac{\nu}{2}}\right]^3(1+|(p_2-q|)^{-(1+\theta_0)}(1+|p_1+q+p_3'|)^{-(1+\theta_0)}(1+|q-p_2'|)^{-(1+\theta_0)}$$

may be estimated by (6.14). In order to show that (6.14) is also an estimating function for the numerator of the second term, it is sufficient to apply *the elementary inequality*

$$(1+|k|)^{-\theta}(1+|k'|)^{-\theta} \leqslant C(1+|k-k'|)^{-\theta}$$

to estimate the product

$$\varphi_{31}\left(p_1+\frac{m_1}{m_3+m_1}q\right)\overline{\varphi_{31}\left(-p_3'-\frac{m_3}{m_3+m_1}q\right)},$$

appearing in this numerator.

We turn to the estimation of the integral (6.8). Let $\eta(q^2)$ be a cutoff function

$$\eta(q^2)=\begin{cases}1 & q^2 \leqslant 4M|z|+1\\ 0 & q^2 \geqslant 4M|z|+2;\end{cases}$$

$$0 \leqslant \eta(q^2) \leqslant 1; \quad |\eta'(q^2)| \leqslant C.$$

We split $\mathcal{F}$ into the sum

$$\mathcal{F}=\int\eta(q^2)\{\ldots\}\,dq+\int[1-\eta(q^2)]\{\ldots\}\,dq=\mathcal{F}_1+\mathcal{F}_2.$$

We do not write here any of the arguments, and the dots in the braces stand for the whole integrand of (6.8).

The estimation of the kernel $\mathcal{F}_2$ presents no difficulty. By (6.11)-(6.13) the denominators are nonsingular within the domain of integration, and we have

$$|\mathcal{F}_2(k, p; k', p'; z)| \leqslant C(1+|z|)^{\frac{3\nu}{2}}(1+|p_1-p_1'|)^{-(1+\theta_0)} \times$$
$$\times \int (1+p_1^2+q^2)^{-1}\left(1+q^2+p_3'^2\right)^{-1}(1+|p_2-q|)^{-(1+\theta_0)} \times$$
$$\times [(1+|p_1+q+p_3'|)^{-(1+\theta_0)}+(1+|q-p_2'|)^{-(1+\theta_0)}]\,dq \leqslant$$
$$\leqslant C(1+|z|)^{\frac{3\nu}{2}}(1+|p_1-p_1'|)^{-(1+\theta_0)}(1+p_1^2+p_3'^2)^{-1} \times$$
$$\times \int (1+q^2)^{-1}(1+|p_2-q|)^{-(1+\theta_0)}[(1+|p_1+q+p_3'|)^{-(1+\theta_0)}+$$
$$+(1+|q-p_2'|)^{-(1+\theta_0)}]\,dq.$$

Here we applied the obvious inequality

$$(1+a^2+x^2)^{-1}(1+x^2+b^2)^{-1} \leqslant (1+a^2+b^2)^{-1}(1+x^2)^{-1}.$$

The integrals on the right-hand side have the form

$$I(a, b) = \int (1+q^2)^{-1}(1+|q-a|)^{-\theta}(1+|q-b|)^{-\theta}dq.$$

These integrals are shown in Appendix II to satisfy the estimate $I(a, b)$:

$$|I(a, b)| \leqslant C(1+|a|)^{-\theta}, \quad \theta < 2. \tag{6.15}$$

Thus we finally obtain for $\mathcal{F}_2$

$$|\mathcal{F}_2(k, p; k', p'; z)| \leqslant$$
$$\leqslant C(1+|z|)^A(1+|p_1-p_1'|)^{-(1+\theta_0)}(1+|p_2|)^{-(1+\theta_0)}(1+p_1^2+p_3'^2)^{-1}, \tag{6.16}$$

where $A = \frac{3\nu}{2}$. In the following sections we shall constantly deal with estimating functions which increase as a power of $|z|$ as $|z| \to \infty$. In all such cases, we write in the corresponding expressions the factor $(1+|z|)^A$, without specifying each time the value of A.

The Hölder differences of the kernel $\mathcal{F}_2(k, p; k', p'; z)$, with index ν, with respect to all the arguments, also satisfy the estimate (6.16). We finally conclude that the kernel $\mathcal{F}_2(k, p; k', p'; z)$ is a Hölder function of all its variables with index ν and the estimating function

$$(1+|z|)^A(1+|p_1-p_1'|)^{-(1+\theta_0)}(1+|p_2|)^{-(1+\theta_0)}\left(1+p_1^2+p_3'^2\right)^{-1}. \tag{6.17}$$

Let us now consider the kernel $\mathcal{F}_1(k, p; k', p'; z)$. The estimates for this kernel are considerably simplified when p_1 or p_3' are sufficiently large. Let, for example, $p_1^2 \geqslant 4M|z|+1$ and $p_3'^2 \leqslant 4M|z|+1$. Then the first denominator in (6.8) is nonsingular. The product of the other two denominators in the second term of (6.8) may be written in the form

$$\frac{1}{\frac{1}{2m_{31}}\left(p_3'^2+\frac{m_3}{m_3+m_1}q\right)^2+\varkappa_{31}^2} \times$$
$$\times \left[\frac{1}{\frac{1}{2m_{12}}\left(q+\frac{m_2}{m_1+m_2}p_3'\right)^2+\frac{p_3'^2}{2n_3}-z}+\frac{1}{z+\varkappa_{31}^2-\frac{q^2}{2n_2}}\right]. \tag{6.18}$$

The first factor is uniformly bounded. We have thus reduced here the integral (6.8) to a sum of integrals with a single denominator, i. e., to ordinary singular integrals.

As an estimating function for the numerator we may take

$$(1+|z|)^A(1+|p_1-p_1'|)^{-(1+\theta_0)}(1+|p_2|)^{-(1+\theta_0)}.$$

This follows from (6. 14) seeing that $|q|$ has in the domain of integration a bound of the order $(1+|z|)^{1/2}$. In the integrals containing $\left[\frac{1}{2m_{12}}\left(q+\frac{m_2}{m_1+m_2}p_3'\right)^2+\right.$ $\left.+\frac{p_3'^2}{2n^3}-z\right]^{-1}$, the integration variable may be changed by substituting

$$q' = q + \frac{m_2}{m_1+m_2}p_3'$$

where $|q'|$ is by definition bounded by a quantity of the order $(1+|z|)^{1/2}$, like $|q|$. By the lemma on singular integrals, the kernel $\mathcal{T}(k, p; k', p'; z)$ is for the p_1 and p_3' in question, a Hölder function of all its arguments with an index ν' that is less than but can be as close as desired to ν, and having the estimating function

$$(1+|z|)^4(1+|p_1-p_1'|)^{-(1+\theta_0)}(1+|p_2|)^{-(1+\theta_0)}(1+p_1^2)^{-1}. \qquad (6.19)$$

The analogous statement for the case $p_1^2 \leqslant 4M|z|+1$, $p_3'^2 \geqslant 4M|z|+1$ is proved in exactly the same way, with $(1+p_1^2)^{-1}$ in (6.19) being replaced by $\left(1+p_3'^2\right)^{-1}$. The case $p_1^2 \geqslant 4M|z|+1$ and $p_3'^2 \geqslant 4M|z|+1$ is even simpler. The estimating function is then (6.17). Note that if $p_3'^2 \leqslant 4M|z|+1$, then

$$(1+p_1^2)^{-1} \leqslant C(1+|z|)(1+p_1^2)^{-1}\left(1+p_3'^2\right)^{-1} \leqslant C(1+|z|)\left(1+p_1^2+p_3'^2\right)^{-1}$$

and similarly when $p_1^2 \leqslant 4M|z|+1$,

$$\left(1+p_3'^2\right)^{-1} \leqslant C(1+|z|)\left(1+p_1^2+p_3'^2\right)^{-1},$$

so that (6.17) is seen to be an estimating function for the kernel $\mathcal{T}_1$ in all the three cases considered.

It is more difficult to investigate the case when both variables p_1 and p_3' are small, or to be exact, when $p_1^2 \leqslant 4M|z|+1$ and $p_3'^2 \leqslant 4M|z|+1$. Then it is impossible to reduce the integral (6.8) to ordinary singular integrals, and we proceed as follows. We rewrite the denominator in the second term of (6.8) using the elementary identity

$$\frac{1}{A}\frac{1}{a}\frac{1}{B} = \frac{1}{A+a}\frac{1}{A}\frac{1}{B} + \frac{1}{A+a}\frac{1}{B+a}\left(\frac{1}{a}+\frac{1}{B}\right), \qquad (6.20)$$

taking for A, a and B the first, second and third denominators. The second term here represents a sum of terms with a single singular denominator. In the first term we have a product of the same two singular denominators as in the first term of (6.8).

The following function may serve as the estimating function for the numerator

$$(1+|z|)^4(1+|p_2-p_2'|)^{-(1+\theta_0)}. \qquad (6.21)$$

The integral of that part of the second term in (6.8) which corresponds to the second term in (6.20), is a sum of ordinary singular integrals, and we deduce, by the lemma on singular integrals, that this term contributes to $\mathcal{T}_1$ a Hölder function with the estimating function (6.21).

The remaining contribution to $\mathcal{T}_1$ is given by the integral

$$\int\left[\frac{1}{2m_{23}}\left(q+\frac{m_2}{m_2+m_3}p_1\right)^2+\frac{p_1^2}{2n_1}-z\right]^{-1} f(k, p; k', p'; q, z)\times$$
$$\times\left[\frac{1}{2m_{12}}\left(q+\frac{m_2}{m_1+m_2}p_3'\right)^2+\frac{p_3'^2}{2n_3}-z\right]^{-1} dq,$$

where the numerator of the integrand is a Hölder function with index ν and the estimating function (6.21). This function falls off smoothly outside the sphere $q^2 = 4M|z| + 1$. If we put

$$a = -\frac{m_2}{m_2 + m_3} p_1; \qquad b = -\frac{m_2}{m_1 + m_2} p_3';$$

$$\xi = 2m_{23}\left(z - \frac{p_1^2}{2n_1}\right); \qquad \eta = 2m_{12}\left(z - \frac{p_3'^2}{2n_3}\right),$$

then this integral assumes the form

$$I(a, b, \xi, \eta) = \int \frac{f(q)\,dq}{[(q-a)^2 - \xi][(q-b)^2 - \eta]}. \tag{6.22}$$

It is easily verified that under our conditions the variables a, b, ξ, η all range over finite regions of the three-dimensional space and the complex plane Π_0, such that a^2, b^2, $|\xi|$, $|\eta|$ are estimated by $C(1+|z|)$ with some constant C.

Integrals of the type (6.22) are studied in detail in Appendix III, where the following result is proved:

Lemma 6.2. Let $f(q)$ be a Hölder function with index μ and such that

$$f(q) = 0, \qquad q^2 \geqslant R > 1.$$

Then the integral $I(a, b, \xi, \eta)$ may be represented in the form

$$I(a, b, \xi, \eta) = \pi |a-b|^{-1} \Bigg\{ \mathcal{T}(a, b, \xi, \eta) +$$

$$+ \int_{-\infty}^{\infty} \frac{r\,dr}{r^2 - \eta} f\left(r\frac{a-b}{|a-b|} + b\right) \ln[(r + |a-b|)^2 - \xi] \Bigg\}, \tag{6.23}$$

where for $a^2 \leqslant CR$, $b^2 \leqslant CR$, $|\xi| \leqslant CR$, $|\eta| \leqslant CR$, $\mathcal{T}(a, b, \xi, \eta)$ is a Hölder function of all its variables with index $\frac{\mu}{4}$ and the estimating function

$$R^{\mu} |a - b - \theta h|^{-\mu} \|f\|_{\mu}. \tag{6.24}$$

Here $\ln(-z)$ stands for that branch of the logarithm which is single-valued on the plane Π_0 slit along the positive real axis and which satisfies the condition

$$\operatorname{Im} \ln(-\omega^2 + i0) = -\operatorname{Im} \ln(-\omega^2 - i0).$$

We have also indicated by $a + \theta h$ the point with the Cartesian coordinates $a_1 + \theta_1 h_1$, $a_2 + \theta_2 h_2$, $a_3 + \theta_3 h_3$, where a_1, a_2, a_3, h_1, h_2, h_3 are the Cartesian coordinates of the points a and h, and θ_1, θ_2, θ_3 are certain numbers, such that $0 \leqslant \theta_i \leqslant 1$, $i = 1, 2, 3$. The function $|a - b - \theta h|^{-\mu}$ may be only called an estimating function with a certain qualification, since it is singular and depends explicitly on h. In this section we shall allow such estimating functions, which provides a short cut that spares us the trouble of writing out several estimates. To this end we agree to write simply $|a + \theta h|^{-\mu}$ whenever the estimating function contains a product of several factors

$$|a + \theta_1 h|^{-\mu_1} |a + \theta_2 h|^{-\mu_2} \ldots |a + \theta_n h|^{-\mu_n},$$

where the sum of exponents $\mu_1, \ldots, \mu_n$ does not exceed μ. Further, $\|f\|_{\mu}$ in (6.24) is defined as

$$\|f\|_{\mu} = \sup_{k, h} \left\{ |f(k)| + \frac{|f(k+h) - f(k)|}{|h|^{\mu}} \right\}.$$

We deduce from Lemma 6.2 that the contribution to the kernel $\mathcal{F}_1(k, p; k', p'; z)$ that contains secondary singularities is given by

$$\mathcal{F}'_1(k, p; k', p'; z) = \left| \frac{m_2}{m_2+m_3} p_1 - \frac{m_2}{m_1+m_2} p'_3 \right|^{-1} \Bigg\{ \mathcal{T}_2(k, p; k', p', z) +$$

$$+ \int\limits_{-\infty}^{\infty} \frac{r\,dr}{\frac{r^2}{2m_{23}} + \frac{p_1^2}{2n_1} - z} \ln \left[\frac{1}{2m_{12}} \left(r + \left| \frac{m_2}{m_2+m_3} p_1 - \frac{m_2}{m_1+m_2} p'_3 \right| \right)^2 + \frac{p_3'^2}{2n_3} - z \right] \times$$

$$\times \mathcal{T}_1(k, p; k', p'; z, r),$$

where $\mathcal{T}_1(k, p; k', p'; z, r)$ and $\mathcal{T}_2(k, p; k', p'; z)$ are Hölder functions of their variables, with the indices ν and $\frac{\nu}{4}$ and the estimating functions $(1+z)^4$ and $(1+z)^4 \left| \frac{m_2}{m_2+m_3} p_1 - \frac{m_2}{m_1+m_2} p'_3 + \theta h \right|^{-\mu}$, respectively. The kernel $\mathcal{T}_1$ can be expressed as a product of $\hat{t}_\alpha(k_\alpha, k'_\alpha, z)$ and the functions $\varphi_\alpha(k_\alpha)$ with elaborate arguments, which we do not bother to write out.

With this we conclude the investigation of the kernel $\mathcal{F}(k, p; k', p'; z)$. One may obtain estimates for the kernels $\mathcal{G}$, $\tilde{\mathcal{G}}$ and $\mathcal{H}$ from the estimates of $\mathcal{F}$ by substituting in the corresponding estimating functions $k'_{12}=0$; $k_{23}=0$; and simultaneously $k_{23}=0$ and $k'_{12}=0$, respectively. Any kernel $\mathcal{Q}^{(2)}_{\gamma_1, \gamma_2, \gamma_3}$ may be dealt with in the same way as $\mathcal{Q}^{(2)}_{23, 31, 12}$.

We denote by $\mathbf{Q}^{(n)}_{\alpha\beta}(z)$ the sum of all the operators $\mathbf{Q}^{(n)}_{\gamma_1, \ldots, \gamma_{n+1}}(z)$ with first and last indices fixed: $\gamma_1 = \alpha$ and $\gamma_{n+1} = \beta$, viz.,

$$\mathbf{Q}^{(n)}_{\alpha\beta}(z) = \sum_{\gamma_1, \ldots, \gamma_{n-1}}{}' \mathbf{Q}^{(n)}_{\alpha, \gamma_1, \ldots, \gamma_{n-1}, \beta}(z).$$

Summation is carried out over the allowed values of $\gamma_1, \ldots, \gamma_{n-1}$. This sum consists, in the case $n=2$, of one term if $\alpha \neq \beta$, and of two terms if $\alpha = \beta$. We have seen on the example of $\mathcal{Q}^{(2)}_{23, 31, 12}$ that the kernels $\mathcal{Q}^{(2)}_{\alpha\beta}$ possess the properties summed up in the following lemma.

Lemma 6.3. The kernels $\mathcal{Q}^{(2)}_{\alpha\beta}(k, p; k', p'; z)$ are of the type $\mathfrak{Q}_{\alpha\beta}$. If $p_\alpha^2 \geqslant 4M|z|+1$ or $p_\beta'^2 \geqslant 4M|z|+1$, then the components $\mathcal{F}^{(2)}_{\alpha\beta}(k, p; k', p'; z)$ of these kernels are Hölder functions of all their variables with index $\nu' < \nu$ and have the estimating functions

$$(1+|z|)^4 (1+|p_\alpha - p'_\alpha|)^{-(1+\theta_0)} \sum_{\gamma \neq \alpha} (1+|p_\gamma|)^{-(1+\theta_0)} \left(1 + p_\alpha^2 + p_\beta'^2\right)^{-1}. \qquad (6.25)$$

The components $\mathcal{G}^{(2)}_{\alpha\beta}$, $\tilde{\mathcal{G}}^{(2)}_{\alpha\beta}$ and $\mathcal{H}^{(2)}_{\alpha\beta}$ possess similar properties, and their estimating functions are obtained from (6.25) by substituting there in turn $k'_\beta=0$; $k_\alpha=0$; and simultaneously, $k_\alpha=0$ and $k'_\beta=0$. These components have secondary singularities when $p'_\alpha \leqslant 4M|z|+1$ and $p'_\beta \leqslant 4M|z|+1$.

We next take up the kernels of the operators $\mathbf{Q}^{(3)}_{\alpha\beta}(z)$. Consider, say, the kernel of $\mathbf{Q}^{(3)}_{12, 23, 31, 12}(z)$. This kernel will be denoted by $\mathcal{Q}^{(3)}$, and the one examined above, by $\mathcal{Q}^{(2)}$. The corresponding components will be correspondingly labelled. The kernel $\mathcal{Q}^{(3)}$ is of the type $\mathfrak{Q}_{12, 12}$. Its component $\mathcal{F}^{(3)}(k, p; k', p'; z)$ has the following representation

$$\mathcal{F}^{(3)}(k, p; k', p'; z) = \int \hat{t}_{12}\left(k_{12}, k''_{12}, z - \frac{p_3^2}{2n_3}\right) \frac{\delta(p_3 - p''_3)}{\frac{k''^2}{2m} + \frac{p''^2}{2n} - z} \times$$

$$\times \left[\mathcal{F}^{(2)}(k'', p''; k', p'; z) + \frac{\varphi_{23}(k''_{23}) \mathcal{F}^{(2)}(p''_1; k', p'; z)}{z + \varkappa_{23}^2 - \frac{p_1''^2}{2n_1}} \right] dk''dp''. \qquad (6.26)$$

We take $p_1''=q$ and p_3'' as integration variables. The integral with respect to the latter variable involves a δ-function and is directly evaluated. We once again represent $\mathscr{F}^{(3)}$ with the help of the step function $\eta(q^2)$ as a sum of two terms

$$\mathscr{F}^{(3)}=\int \eta(q^2)\{\ldots\}\,dq+\int[1-\eta(q^2)]\{\ldots\}\,dq=\mathscr{F}_1^{(3)}+\mathscr{F}_2^{(3)}.$$

The denominators in the integral of $\mathscr{F}_2^{(3)}$ are not singular, and the numerator is a Hölder function with index $\nu'<\nu$ and the estimating function

$$\begin{gathered}(1+|p_1-q|)^{-(1+\theta_0)}(1+|q-p_1'|)^{-(1+\theta_0)}(1+|q+p_3'|)^{-(1+\theta_0)}\leqslant \\ \leqslant(1+|p_2|)^{-(1+\theta_0)}[(1+|p_1-q|)^{-(1+\theta_0)}+(1+|q+p_3|)^{-(1+\theta_0)}]\times \\ \times(1+|q-p_1'|)^{-(1+\theta_0)}.\end{gathered}$$

Using this fact, the estimate (6.15), and the definition of $N(k, p; \theta)$ we deduce that the kernel $\mathscr{F}_2^{(3)}$ is a Hölder function of its variables with index $\nu'<\nu$ and the estimating function

$$(1+|z|)^4 N(k, p; \theta_0)\left(1+p_3^2+p_3'^2\right)^{-1}. \qquad (6.27)$$

Let us now consider the kernel $\mathscr{F}_1^{(3)}(k, p; k', p'; z)$. The kernels $\mathscr{F}^{(2)}$ and $\mathscr{G}^{(2)}$ in the integrand of (6.26) are Hölder functions when $p_3'^2\geqslant 4M|z|+1$. The singular denominators in the second term of (6.26) may be written in a different way, as in (6.18). The following function may serve to estimate the numerator

$$(1+|z|)^4(1+|p_1|)^{-(1+\theta_0)}(1+|p_3|)^{-(1+\theta_0)}\leqslant(1+|z|)^4 N(k, p; \theta_0),$$

since $|q|$ is of the order of $(1+|z|)^{1/4}$ within the domain of integration. We conclude from the lemma on singular integrals that $\mathscr{F}_1^{(3)}$ is in our case a Hölder function with index $\nu'<\nu$ and the estimating function (6.27).

For the case $p_3^2\geqslant 4M|z|+1$ we must use another representation for $\mathscr{F}^{(3)}$ which is obtained by expressing this kernel in terms of the kernels $\mathscr{F}^{(2)}_{12,\,23,\,31}$, $\mathscr{G}^{(2)}_{12,\,23,\,31}$ and $\hat{t}_{12}$, instead of $\hat{t}_{12}$, $\mathscr{F}^{(2)}_{23,\,31,\,12}$ and $\mathscr{G}^{(2)}_{23,\,31,\,12}$ as in (6.26). The case in which both $p_3^2\geqslant 4M|z|+1$ and $p_3'^2\geqslant 4M|z|+1$ is still simpler. We conclude that if $p_3^2\geqslant 4M|z|+1$ or $p_3'^2\geqslant 4M|z|+1$, then the kernel $\mathscr{F}_1^{(3)}$ possesses the same properties as $\mathscr{F}_2^{(3)}$.

Finally, if $p_3^2\leqslant 4M|z|+1$ and $p_3'^2\leqslant 4M|z|+1$, then $\mathscr{F}_1^{(3)}$ has secondary singularities. We shall not describe them here in detail.

Any component of any kernel $\mathscr{F}^{(3)}(k, p; k', p'; z)$ may be investigated in the same way as $\mathscr{Q}^{(3)}_{\alpha\beta}(k, p; k', p'; z)$. The result is summed up in the following lemma:

Lemma 6.4. The kernels $\mathscr{Q}^{(3)}_{\alpha\beta}(k, p; k', p'; z)$ possess all the properties of the $\mathscr{Q}^{(2)}_{\alpha\beta}(k, p; k', p'; z)$, as stated in Lemma 6.3, with the only exception that instead of (6.25) we now have for the estimating function

$$(1+|z|)^4 N(k, p; \theta_0)\left(1+p_\alpha^2+p_\beta'^2\right)^{-1}. \qquad (6.28)$$

It may be shown, by the same considerations which led from Lemma 6.3 to Lemma 6.4, that a similar proposition holds for any kernel $\mathscr{Q}^{(n)}_{\alpha\beta}$ with finite n. However, we can prove that from $n=4$ onward the secondary singularities appear no more for small p_α and p_β'.

Let us prove this assertion for, say, the kernel $\mathscr{Q}^{(4)}_{\gamma_1,\ldots,\gamma_5}$ with $\gamma_1=23$, $\gamma_2=31$, $\gamma_3=12$, $\gamma_4=31$, $\gamma_5=23$. It may be verified on the basis of the preceding discussion that the integral which may contribute secondary singularities *to this kernel has the form*

$$\int\left[\frac{1}{2m_{23}}\left(q_1+\frac{m_2}{m_2+m_3}p_1\right)^2+\frac{p_1^2}{2n_1}-z\right]^{-1}\times$$
$$\times\left[\frac{1}{2m_{12}}\left(q_1+\frac{m_2}{m_1+m_2}q_2\right)^2+\frac{q_2^2}{2n_3}-z\right]^{-1}\times$$
$$\times\,\eta(q_1^2)\,\eta(q_2^2)\,\eta(q_3^2)f(k,\ p;\ k',\ p';\ q_1,\ q_2,\ q_3;\ z)\times$$
$$\times\left[\frac{1}{2m_{12}}\left(q_3+\frac{m_2}{m_1+m_2}q_2\right)^2+\frac{q_2^2}{2n_3}-z\right]^{-1}\times$$
$$\times\left[\frac{1}{2m_{23}}\left(q_3+\frac{m_2}{m_2+m_3}p_1'\right)^2+\frac{p_1'^2}{2n_1}-z\right]^{-1}dq_1dq_2dq_3. \tag{6.29}$$

In the numerator stands a Hölder function of all the variables, having the index ν and the estimating function

$$(1+|z|)^A(1+|p_2-q_1|)^{-(1+\theta_0)}(1+|q_1+p_1+q_2|)^{-(1+\theta_0)}\leqslant$$
$$\leqslant C(1+|z|)^A(1+|p_1|)^{-(1+\theta_0)}(1+|p_2|)^{-(1+\theta_0)}\leqslant$$
$$\leqslant C(1+|z|)^A N(k,\ p;\ \theta_0)(1+p_1^2+p_1'^2)^{-1}.$$

The factor $(1+p_1^2+p_1'^2)^{-1}$ appears on the right-hand side of this estimate since in our case p_1^2 and $p_1'^2$ are of the order of $1+|z|$.

We may first integrate (6.29) with respect to q_1 and q_3. The resulting integrals have the form (6.22). Applying Lemma 6.2 twice, we find that (6.29) becomes

$$\int\eta(q^2)dq\left|\frac{m_2}{m_1+m_2}q-\frac{m_2}{m_2+m_3}p_1\right|^{-1}\left|\frac{m_2}{m_1+m_2}q-\frac{m_2}{m_2+m_3}p_1'\right|^{-1}\times$$
$$\times\left\{\mathcal{T}_{11}(k,\ p;\ k',\ p';\ z,\ q)+\int_{-\infty}^{\infty}\frac{sds}{\frac{s^2}{2m_{23}}+\frac{p_1^2}{2n_1}-z}\times\right.$$
$$\times\ln\left[\frac{1}{2m_{12}}\left(s+\left|\frac{m_2}{m_1+m_2}q-\frac{m_2}{m_2+m_3}p_1\right|\right)^2+\frac{q^2}{2n_3}-z\right]\mathcal{T}_{01}(k,p;k',p';q,z,s)+$$
$$+\int_{-\infty}^{\infty}\frac{rdr}{\frac{r^2}{2m_{23}}+\frac{p_1'^2}{2n_1}-z}\ln\left[\frac{1}{2m_{12}}\left(r+\left|\frac{m_2}{m_1+m_2}q-\frac{m_2}{m_2+m_3}p_1'\right|\right)^2+\frac{q^2}{2n_3}-z\right]\times$$
$$\times\mathcal{T}_{10}(k,\ p;\ k',\ p';\ q,\ z,\ r)+\int_{-\infty}^{\infty}\frac{sds}{\frac{s^2}{2m_{23}}+\frac{p_1^2}{2n_1}-z}\int_{-\infty}^{\infty}\frac{rdr}{\frac{r^2}{2m_{23}}+\frac{p_1'^2}{2n_1}-z}\times$$
$$\times\ln\left[\frac{1}{2m_{12}}\left(s+\left|\frac{m_2}{m_1+m_2}q-\frac{m_2}{m_2+m_3}p_1\right|\right)^2+\frac{q^2}{2n_3}-z\right]\times$$
$$\times\ln\left[\frac{1}{2m_{12}}\left(r+\left|\frac{m_2}{m_1+m_2}q-\frac{m_2}{m_2+m_3}p_1'\right|\right)^2+\frac{q^2}{2n_3}-z\right]\times$$
$$\left.\times\,\mathcal{T}_{00}(k,\ p;\ k',\ p';\ z,\ q,\ s,\ r)\right\},$$

where the kernels $\mathcal{T}_{00}$, $\mathcal{T}_{10}$, $\mathcal{T}_{01}$ and $\mathcal{T}_{11}$ are Hölder functions of their variables. Their estimating functions contain, beside the common factor

$$(1+|z|)^A N(k,\ p;\ \theta_0)(1+p_1^2+p_1'^2)^{-1},$$

the common factor

$$\left|\frac{m_2}{m_1+m_2}q-\frac{m_2}{m_2+m_3}p_1'+\theta h\right|^{-\mu},\quad\left|\frac{m_2}{m_1+m_2}q-\frac{m_2}{m_2+m_3}p_1+\theta h\right|^{-\mu}$$

and

$$\left\{\left|\frac{m_2}{m_1+m_2}q-\frac{m_2}{m_2+m_3}p_1'+\theta_1 h\right|\ \left|\frac{m_2}{m_1+m_2}q-\frac{m_2}{m_2+m_3}p_1+\theta_2 h\right|\right\}^{-\mu},$$

respectively. The last integrand is seen to possess removable singularities, which disappear upon integration with respect to q, which leaves us with ordinary singular integrals with respect to s and r, having Hölder functions in the integrand.

The result is that there exists a certain index $\tilde{\mu}<\frac{\nu}{4}$, such that the integral (6.29) is a Hölder function of k, p, k', p', z having this index and the estimating function (6.28). The same analysis may be carried out for any kernel $\mathcal{Q}^{(4)}_{\gamma_1,\dots,\gamma_s}$.

The kernels $\mathcal{Q}^{(n)}_{\alpha\beta}$ for $n>4$ may be successively reduced to ordinary singular integrals, containing the kernels $\mathcal{Q}^{(4)}_{\alpha\beta}$ in their integrands. Secondary singularities do not appear in the estimation of these integrals.

Let us now state our main result.

Theorem 6.1. *Let the conditions* A_{θ_0}, B_{μ_0}, R *and* C *be fulfilled for the three potentials* $v_\alpha(k)$, $\alpha=23, 31, 12$. *Then the kernels* $\mathcal{Q}^{(n)}_{\alpha\beta}(k, p; k', p'; z)$ *of the operators* $\mathbf{Q}^{(n)}_{\alpha\beta}(z)$ *are of the type* $\mathfrak{Q}_{\alpha\beta}$; *the components* $\mathcal{F}^{(n)}_{\alpha\beta}$, $\mathcal{G}^{(n)}_{\alpha\beta}$, $\tilde{\mathcal{G}}^{(n)}_{\alpha\beta}$ *and* $\mathcal{H}^{(n)}_{\alpha\beta}$ *of these kernels are, for* $n\geqslant 4$, *Hölder functions with some index* $\tilde{\mu}<\frac{1}{8}$ *and have the estimating functions*

$$(1+|z|)^A N(k, p; \tilde{\theta})(1+p_\alpha^2+p_\beta'^2)^{-1};$$

$$(1+|z|)^A(1+|p_\alpha|)^{-2(1+\tilde{\theta})}(1+p_\alpha^2+p_\beta'^2)^{-1},$$

where $\tilde{\theta}$ *may be as close to* θ_0 *from below as we like. The index* $\tilde{\mu}$, *broadly speaking, decreases, while A increases with increasing* n.

We conclude this section with a proof of the statement that the components $\mathcal{F}^{(n)}_{\alpha\beta}$, $\mathcal{G}^{(n)}_{\alpha\beta}$, $\tilde{\mathcal{G}}^{(n)}_{\alpha\beta}$ and $\mathcal{H}^{(n)}_{\alpha\beta}$ of any kernel $Q^{(n)}_{\alpha\beta}$, $n\geqslant 1$ have no secondary singularities when z varies in the neighborhood of the point $z=-\varkappa_\beta^2+\frac{p_\beta'^2}{2n_\beta}$. Let us denote by $\mathcal{F}'^{(n)}_{\alpha\beta}(k, p; k', p'; \xi)$ the kernel

$$\mathcal{F}'^{(n)}_{\alpha\beta}(k, p; k', p'; \xi)=\mathcal{F}^{(n)}_{\alpha\beta}\left(k, p; k', p'; -\varkappa_\beta^2+\frac{p_\beta'^2}{2n_\beta}+\xi\right)$$

and define similarly the kernels $\mathcal{G}'^{(n)}_{\alpha\beta}$, $\tilde{\mathcal{G}}'^{(n)}_{\alpha\beta}$ and $\mathcal{H}'^{(n)}_{\alpha\beta}$. Our contention is precisely formulated in the following lemma.

Lemma 6.5. *Let* $|\xi|\leqslant\frac{1}{2}\bar{\varkappa}^2$, *where* $\bar{\varkappa}^2=\min\limits_\alpha(\varkappa_\alpha^2)$ *and* $\operatorname{Im}\xi\geqslant 0$ *or* $\operatorname{Im}\xi\leqslant 0$. *Then the kernels* $\mathcal{F}'^{(n)}_{\alpha\beta}$, $\mathcal{G}'^{(n)}_{\alpha\beta}$, $\tilde{\mathcal{G}}'^{(n)}_{\alpha\beta}$ *and* $\mathcal{H}'^{(n)}_{\alpha\beta}$ *are Hölder functions of* k, p, k', p' *and* ξ, *with index* $\nu'<\nu$. *The corresponding estimating function is uniformly bounded for arbitrary* k, p, k_β' *and for* p_β' *restricted to a finite region.*

Proof. Consider the case $n=1$. An explicit expression for $\mathcal{Q}^{(1)}_{\alpha\beta}$ with $\alpha=23$ and $\beta=31$ was given before, i. e., (5.1). It is evident from (5.1) that secondary singularities appear in the kernel $\mathcal{F}^{(1)}_{23,31}$ when the expression

$$\frac{p_1^2}{2m_{31}}+\frac{(p_1, p_2')}{m_3}+\frac{p_2'^2}{2m_{23}}-z=\frac{1}{2m_{23}}\left(p_1+\frac{m_1}{m_2+m_3}p_2'\right)^2+\frac{p_2'^2}{2n_2}-z,$$

vanishes. Setting

$$z=-\varkappa_{31}^2+\frac{p_2'^2}{2n_2}+\xi,$$

we obtain the last expression in the form

$$\frac{1}{2m_{23}}\left(p_1+\frac{m_1}{m_2+m_3}p_2'\right)^2+\varkappa_{31}^2+\xi. \tag{6.30}$$

If $|\xi|\leqslant\frac{1}{2}\varkappa_{31}^2$, then (6.30) does not vanish and hence the kernel $\mathcal{T}^{(1)}_{23,\,31}$ is uniformly bounded. The Hölder differences with index ν of this kernel are also uniformly bounded if p_2' is restricted to a finite region.

Any component of any kernel $\mathcal{Q}^{(1)}_{\alpha\beta}$ may be treated analogously.

We have indicated, in the instance of the transition from the kernels $\mathcal{Q}^{(2)}_{\alpha\beta}$ to the kernels $\mathcal{Q}^{(3)}_{\alpha\beta}$, all the steps required in order to construct the components of the kernel $\mathcal{Q}^{(n)}_{\alpha\beta}$ in terms of the components of $\mathcal{Q}^{(n-1)}_{\alpha\beta}$ and t_α. It is obvious from this example that no secondary singularities can arise in $\mathcal{Q}^{(n)}_{\alpha\beta}$ if there are none in $\mathcal{Q}^{(n-1)}_{\alpha\beta}$. In the present case the kernels $\mathcal{Q}^{(1)}_{\alpha\beta}$ have no such singularities, whence it follows, by induction, that there will also be none in $\mathcal{Q}^{(n)}_{\alpha\beta}$ for any $n\geqslant 1$. Consequently, the components of these kernels are Hölder functions of their variables with bounded estimating functions, which proves our lemma.

This result will be applied in § 11.

§ 7. The operator $\mathbf{A}(z)$

This section is devoted to an investigation of the operator $\mathbf{A}(z)$ in the Banach space $\mathfrak{B}(\theta,\mu)$, introduced in § 5. We shall show that $\mathbf{A}(z)$ is defined on a dense set in $\mathfrak{B}(\theta,\mu)$, where μ is less than the ν of Lemma 6.1, which set consists of functions belonging to $\mathfrak{B}(\theta,\mu')$, where μ' is any index $\mu'>\mu$. We shall further show that all the integral powers $\mathbf{A}^n(z)$ of the operator $\mathbf{A}(z)$ are defined on $\mathfrak{B}(\theta,\mu')$, and that for $n\geqslant 5$ the operators $\mathbf{A}^n(z)$ may be extended into completely continuous operators in $\mathfrak{B}(\theta,\mu)$, if $\theta<\bar\theta$ and $\mu<\bar\mu$ ($\bar\theta$ and $\bar\mu$ were defined in § 6). We shall finally study the homogeneous equation $\omega=\mathbf{A}(z)\omega$. The obtained results will serve as a basis for the discussion of the solvability of a second-kind equation with the operator $\mathbf{A}(z)$.

We first examine the domain of $\mathbf{A}(z)$. Typical integrals, to be estimated below, are of the form

$$f_1(k,p,z)=\int\hat t_{23}\left(k_{23},k_{23}',z-\frac{p_1^2}{2n_1}\right)\frac{\delta(p_1-p_1')}{\frac{k'^2}{2m}+\frac{p'^2}{2n}-z}\rho_{31}(k',p')\,dk'dp'; \tag{7.1}$$

$$f_2(k,p,z)=\int\hat t_{23}\left(k_{23},k_{23}',z-\frac{p_1^2}{2n_1}\right)\frac{\delta(p_1-p_1')}{\frac{k'^2}{2m}+\frac{p'^2}{2n}-z}\,\frac{\varphi_{31}(k_{31}')\,\sigma_{31}(p_2')}{z+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}}\,dk'dp'. \tag{7.2}$$

Lemma 7.1. Let the function

$$\tilde\rho_{31}(k,p)=(1+|k_{31}|)^{\delta}\rho_{31}(k,p)$$

belong to the class $\mathfrak{M}(\theta,\mu')$ *and* $\sigma_{31}(p)$ *to the class*. $\mathfrak{N}(\theta,\mu')$, *where*

$$\delta+\theta<1;\ \theta<\theta_0;\ \mu'<\nu.$$

Then for any $z\in\Pi_{-\varkappa^2}$ *the functions*

$$\tilde f_i(k,p)=(1+|k_{23}|)^{\theta_0-\theta}(1+|z|)^{\frac{\delta_1-\mu'}{2}}f_i(k,p),\ i=1,2,$$

belong to the class $\mathfrak{M}(\theta, \mu)$, *where* μ *is smaller than and can be taken as close as desired to* μ', $\theta < \theta_0$ *and* $\delta_1 = \min(\theta, \delta)$.

Proof. Consider first $f_1(k, p)$. We take k'_{23} and p'_1 as the integration variables. The integral with respect to p'_1 is removed due to the δ-function. The integral over k'_{23} is estimated with the help of the lemma on singular integrals (cf. § 4). The integrals over angular variables are estimated by the methods applied in § 4. Note that of the three terms in the estimating function $N(k, p; \theta)$, we only need consider two. Thus, for example,

$$1+|p_2|)^{-1}(1+|p_3|)^{-1} \leqslant C(1+|p_2+p_3|)^{-1}\{(1+|p_2|)^{-1}+(1+|p_3|)^{-1}\} =$$
$$= C(1+|p_1|)^{-1}\{(1+|p_2|)^{-1}+(1+|p_3|)^{-1}\}.$$

The numerator in (7.1) is thus estimated by the function

$$(1+|p_1|)^{-(1+\theta)}(1+|k'_{31}|)^{-\delta}\Big\{(1+|p_2-p'_2|)^{-(1+\theta_0)}(1+|p'_2|)^{-(1+\theta)}+$$
$$+(1+|p_3-p'_3|)^{-(1+\theta_0)}(1+|p'_3|)^{-(1+\theta)}\Big\}. \tag{7.3}$$

Here we made use of the fact that for $p_1 = p'_1$,

$$|k_{23}-k'_{23}| = |p_2-p'_2| = |p_3-p'_3|.$$

The first term in (7.3) may be written

$$M(k, p; k', p') = (1+|p_1|)^{-(1+\theta)}(1+|p_2|)^{-(1+\theta)}(1+|k'_{31}|)^{-\delta}\times$$
$$\times(1+|p_2-p'_2|)^{-(\theta_0-\theta)}\{(1+|p'_2|)^{-(1+\theta)}+(1+|p_2-p'_2|)^{-(1+\theta)}\},$$

and, expressing here p'_2 and k'_{31} in terms of k'_{23} and $p'_1 = p_1$, we obtain the following estimate for the integral of this term over the angular variables

$$\int d\Omega_{k'_{23}} M(k, p; k', p')\big|_{p'_1=p_1} \leqslant$$
$$\leqslant C(1+|p_1|)^{-(1+\theta)-\delta}(1+|p_2|)^{-(1+\theta)}(1+|k_{23}|)^{-(\theta_0-\theta)}(1+|k'_{23}|)^{-(1+\theta)}.$$

The second term is estimated similarly. We recall that the factor $(1+|z|)^{\mu'/2}$ enters, in view of Lemma 6.1, in the estimating function of the kernel $t_{23}\left(k_{23}, k'_{23}, z-\frac{p_1^2}{2n_1}\right)$. We find, by the lemma on singular integrals, that $f_1(k, p, z)$ is a Hölder function of all its variables with index $\mu < \mu'$ and the estimating function

$$N(k, p; \theta)(1+|k_{23}|)^{-(\theta_0-\theta)}(1+|p_1|)^{-\delta}\left(1+\left|z-\frac{p_1^2}{2n_1}\right|\right)^{-\theta/2}(1+|z|)^{\nu/2} \leqslant$$
$$\leqslant CN(k, p; \theta)(1+|z|)^{\frac{\mu'-\delta_1}{2}}(1+|k_{23}|)^{-(\theta_0-\theta)};\ \delta_1 = \min(\theta, \delta). \tag{7.4}$$

Consider now $f_2(k, p)$. We separate the singularities in the denominator of (7.2) as follows

$$\frac{1}{\frac{k'^2}{2m}+\frac{p'^2}{2n}-z}\ \frac{1}{z+\varkappa_{31}^2-\frac{p'^2}{2n_2}} =$$

$$= \frac{1}{\frac{k'^2_{31}}{2m_{31}}+\varkappa_{31}^2}\left(\frac{1}{\frac{k'^2}{2m}+\frac{p'^2}{2n}-z}+\frac{1}{z+\varkappa_{31}^2-\frac{p'^2_2}{2n_2}}\right). \tag{7.5}$$

The integral of the first term in (7.5) is estimated exactly as $f_1(k, p, z)$. One should only keep in mind that the product $\varphi_{31}(k_{31})\sigma_{31}(p_2)$ is estimated by the function

$$(1+|k_{31}|)^{-(1+\theta_0)}(1+|p_2|)^{-2(1+\theta)} \leqslant CN(k, p; \theta)(1+|k_{31}|)^{-(\theta_0-\theta)},$$

i. e., by the same function as $\rho_{31}(k, p)$, where $\delta=\theta_0-\theta$. Therefore, the estimate of the contribution of this term to $f_2(k, p, z)$ coincides with (7.4), with $\delta_1=\min(\theta_0-\theta, \theta)$.

To estimate the integral with the denominator $\left(z+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}\right)^{-1}$, we take p_1' and p_2' as the integration variables. The integral over the angular variables of p_2' is estimated as follows

$$\int d\Omega_{p_2'}(1+|p_2-p_2'|)^{-(1+\theta_0)}(1+|p_2'|)^{-(1+\theta)} \leqslant$$
$$\leqslant C(1+|p_2|)^{-(1+\theta)}(1+|k_{23}|)^{-(\theta_0-\theta)}(1+|p_2'|)^{-(1+\theta)}.$$

By the lemma on singular integrals, the contribution of the second term to $f_2(k, p, z)$ is a Hölder function with the estimating function

$$N(k, p; \theta)(1+|k_{23}|)^{-(\theta_0-\theta)}(1+|z|)^{-\theta/2},$$

which is subordinate to the estimating function in (7.4). This completes the proof.

Note that in the definition of the operator $\mathbf{A}(z)$ there also appear integrals of the type

$$f_3(k, p, z)=\int \overline{\varphi_{23}(k_{32}')}\,\frac{\delta(p_1-p_1')}{\frac{k'^2}{2m}+\frac{p'^2}{2n}-z}\,\rho_{31}(k', p')\,dk'dp'; \tag{7.6}$$

$$f_4(k, p, z)=\int \overline{\varphi_{23}(k_{23}')}\,\frac{\delta(p_1-p_1')}{\frac{k'^2}{2m}+\frac{p'^2}{2n}-z}\,\frac{\varphi_{31}(k_{31}')\,\sigma_{31}(p_2')}{z+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}}\,dk'dp'. \tag{7.7}$$

The integrals (7.6) and (7.7) are estimated in the same way as (7.1) and (7.2), respectively. The estimating function for these integrals may be obtained from (7.4) by replacing the estimating function $N(k, p; \theta)$ on the right-hand side by $(1+|p_1|)^{-2(1+\theta)}$.

The preceding results establish that the integral operator $\mathbf{A}(z)$ is defined in $\mathfrak{B}(\theta, \mu)$ on a dense set, consisting of $\omega\in\mathfrak{B}(\theta, \mu')$, $\mu'>\mu$, and that

$$\|\mathbf{A}(z)\omega\|_{\theta,\mu} \leqslant C(1+|z|)^{\frac{\mu'}{2}}\|\omega\|_{\theta,\mu'}; \tag{7.8}$$

$$\|[\mathbf{A}(z+\Delta)-\mathbf{A}(z)]\omega\|_{\theta,\mu} \leqslant C(1+|z|)^{\frac{\mu'}{2}}\|\omega\|_{\theta,\mu'}|\Delta|^{\mu'-\mu}. \tag{7.9}$$

Incidentally, this result may also be derived with the help of rougher estimates than those of Lemma 7.1. Namely, the not very essential factors $(1+|k_{31}'|)^{\theta}$ and $(1+|k_{23}|)^{-(\theta_0-\theta)}$ may be left out in the condition and in the assertion.

Using the estimates of Lemma 7.1 in their complete form, we can prove

$$\|\mathbf{A}^2(z)\omega\|_{\theta,\mu} \leqslant C(1+|z|)^{\mu'-\frac{\theta_0-\theta}{2}}\|\omega\|_{\theta,\mu'}. \tag{7.10}$$

Indeed, applying twice the operator $\mathbf{A}(z)$ to the element $\omega\in\mathfrak{B}(\theta, \mu')$, we obtain in the first step a factor of the type $(1+|k|)^{-(\theta_0-\theta)}$, and hence in the second step the factor $(1+|z|)^{-\frac{\theta_0-\theta}{2}}$, which furnishes the estimate (7.10).

We finally state that (7.8) holds good if $\mathbf{A}(z)$ is replaced on the left-hand side by any finite power $\mathbf{A}^n(z)$. To see this we only need to apply (7.8) n times.

We have thus arrived at the following result:

Lemma 7.2. The operator $\mathbf{A}(z)$ *for any* z *in* $\Pi_{-\varkappa^2}$ *is defined in* $\mathfrak{B}(\theta, \mu)$, $\theta < \theta_0$, $\mu < \nu$ *on a dense set, consisting of the elements* ω *of* $\mathfrak{B}(\theta, \mu')$, $\mu' > \mu$. *Any finite power* $\mathbf{A}^n(z)$ *is also defined on this set. If the indices* θ *and* μ *satisfy the conditions*

$$\frac{\theta_0-\theta}{2} > \mu; \quad \theta_0 + \frac{\theta_0-\theta}{2} < 1, \tag{7.11}$$

then the ratio $\dfrac{\|\mathbf{A}^2(z)\,\omega\|_{\theta,\mu}}{\|\omega\|_{\theta,\mu'}}$ *becomes as small as desired when* $|z| \to \infty$.

The estimates derived in § 6 for the kernels of the operators $\mathbf{Q}^{(n}_{\gamma_1,\ldots,\gamma_{n+1}}(z)$, enable us to say more about the powers $\mathbf{A}^{(n)}(z)$ for $n \geqslant 5$. Comparing the definitions (5.12), (5.13) of the operator $\mathbf{A}(z)$ and the operators $\mathbf{Q}^{(n)}_{\gamma_1,\ldots,\gamma_{n+1}}(z)$, we conclude that the effect of the operation of $\mathbf{A}^{n+1}(z)$ may be expressed by means of the kernels $\mathscr{Q}^{(n)}_{\gamma_1,\ldots,\gamma_{n+1}}(k, p; k', p'; z)$ as follows: the transformation $\omega' = \mathbf{A}^{n+1}(z)\,\omega$ is equivalent to

$$\rho'_\alpha(k, p) = \int \sum_{\substack{\beta,\gamma\\ \beta\neq\gamma}} \left\{ \left[\mathscr{F}^{(n)}_{\alpha\gamma}(k, p; k', p'; z) + \mathscr{G}^{(n)}_{\alpha\gamma}(k, p; p'_\gamma) \frac{\overline{\varphi_\gamma(k'_\gamma)}}{z + \varkappa^2_\gamma - \dfrac{p'^2_\gamma}{2n_\gamma}} \right] \times \right.$$
$$\left. \times \frac{1}{\dfrac{k'^2}{2m} + \dfrac{p'^2}{2n} - z} \left[\rho_\beta(k', p') + \frac{\varphi_\beta(k'_\beta)\,\sigma_\beta(p'_\beta)}{z + \varkappa^2_\beta - \dfrac{p'^2_\beta}{2n_\beta}} \right] \right\} dk'dp', \tag{7.12}$$

$$\sigma'_\alpha(p_\alpha) = \int \sum_{\substack{\beta,\gamma\\ \beta\neq\gamma}} \left\{ \left[\tilde{\mathscr{G}}^{(n)}_{\alpha\gamma}(p_\alpha; k'\, p'; z) + \mathscr{H}^{(n)}_{\alpha\gamma}(p_\alpha; p'_\beta; z) \frac{\overline{\varphi_\gamma(k'_\gamma)}}{z + \varkappa^2_\gamma - \dfrac{p'^2_\gamma}{2n_\gamma}} \right] \times \right.$$
$$\left. \times \frac{1}{\dfrac{k'^2}{2m} + \dfrac{p'^2}{2n} - z} \left[\rho_\beta(k', p') + \frac{\varphi_\beta(k'_\beta)\,\sigma_\beta(p'_\beta)}{z + \varkappa^2_\beta - \dfrac{p'^2_\beta}{2n_\beta}} \right] \right\} dk'dp'. \tag{7.13}$$

Here we have again written $\mathscr{F}^{(n)}_{\alpha\beta}$, $\mathscr{G}^{(n)}_{\alpha\beta}$, $\tilde{\mathscr{G}}^{(n)}_{\alpha\beta}$ and $\mathscr{H}^{(n)}_{\alpha\beta}$ for the components of the kernel $\mathscr{Q}^{(n)}_{\alpha\beta}(k, p; k', p'; z)$ of the operator $\mathbf{Q}^{(n)}_{\alpha\beta}(z)$

$$\mathbf{Q}^{(n)}_{\alpha\beta}(z) = \sum_{\gamma_1,\ldots,\gamma_{n-1}} \mathbf{Q}^{(n)}_{\alpha,\gamma_1,\ldots,\gamma_{n-1},\beta}(z),$$

where summation is carried out over all the allowed values of $\gamma_1, \ldots, \gamma_{n-1}$.

The changes of order of integration which are required in the derivation of (7.12) and (7.13) are permitted when $\operatorname{Im} z \neq 0$, since the integrals in question converge absolutely. For $\operatorname{Im} z \to 0$ these formulas follow from the continuity in z of all the integrals involved.

Using the estimates derived in § 5 for the kernels $\mathscr{Q}^{(n)}_{\alpha\beta}(k, p; k', p'; z)$, and the lemma on singular integrals, we can prove the following assertion:

Lemma 7.3. Let $\omega \in \mathfrak{B}(\theta, \mu)$ *and* $\theta < \tilde{\theta}$, $\mu < \tilde{\mu}$. *Then the following estimates hold for any* z *in* $\Pi_{-\varkappa^2}$ *and* $n \geqslant 5$

$$\|\mathbf{A}^n(z)\,\omega\|_{\theta',\mu'} \leqslant C(1 + |z|)^4 \|\omega\|_{\theta,\mu}; \tag{7.14}$$

$$\|[\mathbf{A}^n(z + \Delta) - \mathbf{A}^n(z)]\,\omega\|_{\theta',\mu'} \leqslant C(1 + |z|)^4 \|\omega\|_{\theta,\mu} |\Delta|^\delta; \tag{7.15}$$

where μ', θ' *and* δ *are any indices, such that*

$$\mu' < \bar{\mu}; \quad \theta' < \bar{\theta}; \quad \delta < \bar{\mu} - \mu',$$

and A is some fixed number.

We shall not carry out the proof in detail. The tools required for it are all at hand. We have to separate the singularities in the denominators of the integrals, choose suitable integration variables, estimate the integrals over the angle variables of the corresponding estimating functions, and apply the lemma on singular integrals.

We now make use of the fact that a sequence ω_n which is uniformly bounded in $\mathfrak{B}(\theta', \mu')$, is compact in $\mathfrak{B}(\theta, \mu)$ for any $\theta < \theta'$, $\mu < \mu'$. By Lemma 7.3 we then have

Lemma 7.4. The operator $\mathbf{A}^n(z)$ *may be extended from a dense set over the entire space* $\mathfrak{B}(\theta, \mu)$, *with* $\theta < \bar{\theta}$, $\mu < \bar{\mu}$, *and the resulting operator is completely continuous and depends continuously on* z *within any finite region of the complex plane* Π_{-x^2}.

Let us now turn to the homogeneous equation

$$\omega = \mathbf{A}(z)\,\omega. \tag{7.16}$$

Lemma 7.5. Let ω *be a solution of equation (7.16), where* $\omega \in \mathfrak{B}(\theta, \mu)$, $\theta < \bar{\theta}$, $\mu < \bar{\mu}$. *Then* $\omega \in \mathfrak{B}(\theta', \mu')$, $\theta' = \bar{\theta} - \varepsilon$, $\mu' = \bar{\mu} - \varepsilon$, $\varepsilon > 0$.

The proof is based on the fact that the ω which satisfies equation (7.16) will also solve the equation

$$\omega = \mathbf{A}^n(z)\,\omega. \tag{7.17}$$

Lemma 7.5 follows now from Lemma 7.3.

Lemma 7.6. Let $\operatorname{Im} z \neq 0$. *Then equation (7.16) has no nontrivial solutions in* $\mathfrak{B}(\theta, \mu)$ *for any* $\theta < \bar{\theta}$, $\mu < \bar{\mu}$.

Proof. Let ω be a solution of (7.16), belonging to $\mathfrak{B}(\theta, \mu)$. If $\rho_\alpha(k, p)$ and $\sigma_\alpha(p_\alpha)$ are the components of ω, then by Lemma 7.5 $\rho_\alpha(k, p) \in \mathfrak{M}(\bar{\theta} - \varepsilon, \bar{\mu} - \varepsilon)$ and $\sigma_\alpha(p_\alpha) \in \mathfrak{N}(\bar{\theta} - \varepsilon, \bar{\mu} - \varepsilon)$. Since $\bar{\theta} > \frac{1}{2}$, it is evident that the functions $\chi_\alpha(k, p; z)$ associated with $\rho_\alpha(k, p)$ and $\sigma_\alpha(p_\alpha)$ (cf. (5.10)) are square-integrable over the entire space, i. e., $\chi_\alpha(k, p) \in \mathfrak{H}$, and

$$\psi_\alpha(k, p) = \left(\frac{k^2}{2m} = \frac{p^2}{2n} - z\right)^{-1} \chi_\alpha(k, p; z) \in \mathfrak{D}.$$

Equation (7.16) may be written in terms of $\psi_\alpha(k, p)$ as

$$\psi_\alpha = -\mathbf{R}_0(z)\,\mathbf{T}_\alpha(z) \sum_{\beta \neq \alpha} \psi_\beta. \tag{7.18}$$

Writing

$$\psi(k, p) = \psi_{23}(k, p) + \psi_{31}(k, p) + \psi_{12}(k, p).$$

and multiplying (7.18) by $\mathbf{E} + \mathbf{R}_\alpha(z)\mathbf{V}_\alpha$, we obtain, on account of (3.12),

$$\psi_\alpha = -\mathbf{R}_0(z)\,\mathbf{V}_\alpha \psi \tag{7.19}$$

and after summation over α

$$\psi = -\mathbf{R}_0(z)(\mathbf{V}_{23} + \mathbf{V}_{31} + \mathbf{V}_{12})\,\psi. \tag{7.20}$$

Multiplying (7.20) by $\mathbf{H}_0 - z\mathbf{E}$, we find that ψ satisfies the equation

$$\mathbf{H}\psi = z\psi,$$

which has no nontrivial solutions for $\operatorname{Im} z \neq 0$, since $\mathbf{H}$ is self-adjoint. Thus $\psi \equiv 0$, which in view of (7.19) entails $\psi_\alpha = 0$, $\alpha = 23, 31, 12$, and hence $\chi_\alpha(k, p; z) = 0$. It finally follows by definition of $\mathbf{A}(z)$ that $\rho_\alpha(k, p) = 0$ and $\sigma_\alpha(p_\alpha) = 0$, i. e. , $\omega = 0$. This completes the proof.

Lemma 7.7. For sufficiently large $|z|$ *equation (7.16) has no nontrivial solutions.*

Proof. In view of Lemma 7.5 we may assume without loss of generality that $\omega \in \mathfrak{B}(\theta, \mu)$, where $\theta \leqslant \tilde{\theta} - \varepsilon$, $\mu \leqslant \tilde{\mu} - \varepsilon$, and the indices θ and μ may be chosen to satisfy (7.11). We set $n = 2N + 5$ in (7.17) and apply the estimates (7.14) and (7.10), which gives

$$\|\omega\|_{\theta, \mu} \leqslant C \frac{(1 + |z|)^A}{\left[(1 + |z|)^{\frac{\theta_0 - \theta}{2} - \mu}\right]^N} \|\omega\|_{\theta, \mu}. \tag{7.21}$$

The indices θ and μ are chosen so that $\frac{\theta_0 - \theta}{2} - \mu = \delta > 0$. Let us take $N = \left[\frac{A}{\delta}\right] + 1$. The coefficient of $\|\omega\|_{\theta, \mu}$ on the right-hand side of (7.21) becomes arbitrarily small for sufficiently large $|z|$ and hence $\omega = 0$. This completes the proof.

Let $\Phi^{(+)}$ denote the set of real points λ for which equation (7.16) with $z = \lambda + i0$ has a nontrivial solution, and $\Phi^{(-)}$ the analogous set with $z = \lambda - i0$. The points of $\Phi^{(+)}$ and $\Phi^{(-)}$ will be called the singular points of the operator $\mathbf{A}(z)$.

Lemma 7.8. The set $\Phi^{(+)}$ *is closed.*

Proof. Let $\{\lambda_n\}$, $n = 1, 2, \ldots$ be an infinite sequence of singular points belonging to $\Phi^{(+)}$, which converges to the point λ_0. We denote by ω_n the solutions of equation (7.16) with $z = \lambda_n + i0$, normalized by the condition

$$\|\omega_n\|_{\theta, \mu} = 1,$$

where θ, μ satisfy $\theta < \tilde{\theta}$, $\mu < \tilde{\mu}$.

Consider the sequence

$$\tilde{\omega}_n = \mathbf{A}^N(\lambda_0 + i0)\,\omega_n,$$

for $N \geqslant 5$. In virtue of the complete continuity of the operator $\mathbf{A}^N(\lambda_0 + i0)$, it is possible to pick out from $\{\tilde{\omega}_n\}$ a convergent subsequence; let its limit element be ω_0. Since we shall only have to deal with this subsequence, we reserve the symbols $\{\tilde{\omega}_n\}$ and $\{\omega_n\}$ for it and for the corresponding subsequence of ω_n. Equation (7.17), which is satisfied by ω_n with $z = \lambda_n + i0$, shows that we may write

$$\omega_n = \tilde{\omega}_n + [\mathbf{A}^N(\lambda_n + i0) - \mathbf{A}^N(\lambda_0 + i0)]\,\omega_n.$$

The continuity of $\mathbf{A}^N(z)$ in z implies that the ω_n also converge to ω_0. We may therefore pass to the limit in the equation

$$\omega_n = \mathbf{A}(\lambda_n + i0)\,\omega_n$$

and find that ω_0 is a nontrivial solution of (7.16) with $z = \lambda_0 + i0$, that is, $\lambda_0 \in \Phi^{(+)}$, which proves the lemma.

Lemma 7.9. Let $\lambda \in \Phi^{(+)}$ *and let* ω *be a solution of (7.16) with* $z = \lambda + i0$ *and* $\rho_\alpha(k, p)$ *and* $\sigma_\alpha(p_\alpha)$ *the components of the element* ω. *Then*

$$\int |\tilde{\chi}_{23}(k, p) + \tilde{\chi}_{31}(k, p) + \tilde{\chi}_{12}(k, p)|^2\, \delta\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda\right) dk\, dp = 0; \tag{7.22}$$

$$\int |\sigma_\alpha(p_\alpha)|^2 \delta\left(-\varkappa_\alpha^2+\frac{p_\alpha^2}{2n_\alpha}-\lambda\right) dp_\alpha=0; \quad \alpha=23,\ 31,\ 12. \tag{7.23}$$

The function

$$\tilde{\chi}_\alpha(k,\ p)=\rho_\alpha(k,\ p)+\frac{\varphi_\alpha(k_\alpha)\,\sigma_\alpha(p_\alpha)}{\dfrac{k_\alpha^2}{2m_\alpha}+\varkappa_\alpha^2}, \tag{7.24}$$

is the one obtained from $\chi_\alpha(k,\ p;\ z)$ on setting $z=\frac{k^2}{2m}+\frac{p^2}{2n}$.

The proof resembles that of Lemma 4.7 but is rather more involved. We construct from $\rho_\alpha(k,\ p)$ and $\sigma_\alpha(p_\alpha)$ the functions

$$\chi_\alpha(k,\ p;\ \lambda+i\varepsilon)=\rho_\alpha(k,\ p)+\frac{\varphi_\alpha(k_\alpha)\,\sigma_\alpha(p_\alpha)}{\lambda+i\varepsilon+\varkappa_\alpha^2-\dfrac{p_\alpha^2}{2n_\alpha}} \tag{7.25}$$

and consider the functions

$$\hat{\chi}_\alpha(k,\ p;\ \varepsilon)=-\int t_\alpha\left(k_\alpha,\ k'_\alpha,\ \lambda+i\varepsilon-\frac{p_\alpha^2}{2n_\alpha}\right)\frac{\delta(p_\alpha-p'_\alpha)}{\dfrac{k'^2}{2m}+\dfrac{p'^2}{2n}-z}\times$$

$$\times\sum_\beta \chi_\beta(k',\ p';\ \lambda+i\varepsilon)\,dk'dp'. \tag{7.26}$$

These $\hat{\chi}_\alpha(k,\ p;\ \varepsilon)$ may be represented in the form (7.25), with $\rho_\alpha(k,\ p;\ \varepsilon)$ and $\sigma_\alpha(p_\alpha;\ \varepsilon)$ as the components of the element $\omega(\varepsilon)$. Thus (7.26) becomes

$$\omega(\varepsilon)=\mathbf{A}(\lambda+i\varepsilon)\,\omega=\omega+[\mathbf{A}(\lambda+i\varepsilon)-\mathbf{A}(\lambda+i0)]\,\omega,$$

and now it follows from Lemma 7.2 that for $\varepsilon\to+0$

$$\|\omega(\varepsilon)-\omega\|_{\theta,\ \mu}=o(1),$$

where θ, μ satisfy $\theta<\tilde{\theta}$, $\mu<\tilde{\mu}$.

If $\varepsilon\neq 0$, then the functions $\chi_\alpha(k,\ p;\ \lambda+i\varepsilon)$ and $\hat{\chi}_\alpha(k,\ p;\ \varepsilon)$ are square-integrable over the entire space, and (7.26) may be written as

$$\hat{\chi}_\alpha=-\mathbf{T}_\alpha(\lambda+i\varepsilon)\,\mathbf{R}_0(\lambda+i\varepsilon)\sum_{\beta\neq\alpha}\chi_\beta. \tag{7.27}$$

Here $\hat{\chi}_\alpha$ and χ_α are elements of $\mathfrak{H}$, defined by the functions $\hat{\chi}_\alpha(k,\ p;\ \varepsilon)$ and $\chi_\alpha(k,\ p;\ \lambda+i\varepsilon)$, respectively.

Multiplying (7.27) by $\mathbf{E}+\mathbf{V}_\alpha\mathbf{R}_0(\lambda+i\varepsilon)$ and applying (3.12), we obtain

$$\hat{\chi}_\alpha=-\mathbf{V}_\alpha\mathbf{R}_0(\lambda+i\varepsilon)\,\hat{\chi}_\alpha-\mathbf{V}_\alpha\mathbf{R}_0(\lambda+i\varepsilon)\sum_{\beta\neq\alpha}\chi_\beta. \tag{7.28}$$

Scalar multiplication of (7.28) on the left and right, first by $\mathbf{R}_0(\lambda+i\varepsilon)\,\hat{\chi}_\alpha$, and then again by $\mathbf{R}_0(\lambda+i\varepsilon)\sum\limits_{\beta\neq\alpha}\chi_\beta$, gives, after combining the results and in view of the symmetry of the operators $\mathbf{V}_\alpha$ and $\mathfrak{D}$,

$$([\mathbf{R}_0(\lambda+i\varepsilon)-\mathbf{R}_0(\lambda-i\varepsilon)]\,\hat{\chi}_\alpha,\ \hat{\chi}_\alpha)+\left(\mathbf{R}_0(\lambda+i\varepsilon)\sum_{\beta\neq\alpha}\chi_\beta,\ \hat{\chi}_\alpha\right)-$$

$$-\left(\hat{\chi}_\alpha,\ \mathbf{R}_0(\lambda+i\varepsilon)\sum_{\beta\neq\alpha}\chi_\beta\right)=0. \tag{7.29}$$

Let us consider one typical integral which contributes to the second and third terms of (7.29):

$$\int\left\{\rho_\alpha(k,\ p;\ \varepsilon)+\frac{\varphi_\alpha(k_\alpha)\,\sigma_\alpha(p_\alpha;\ \varepsilon)}{\lambda+i\varepsilon+\varkappa_\alpha^2-\dfrac{p_\alpha^2}{2n_\alpha}}\right\}\frac{1}{\dfrac{k^2}{2m}+\dfrac{p^2}{2n}-\lambda+i\varepsilon}\left\{\overline{\rho_\beta(k,\ p)}+\frac{\overline{\varphi_\beta(k_\beta)}\,\overline{\sigma_\beta(p_\beta)}}{\lambda-i\varepsilon+\varkappa_\beta^2-\dfrac{p_\beta^2}{2n_\beta}}\right\}dkdp.$$

All the singularities of the denominators in the integrand disappear when $\alpha \neq \beta$, and the integral depends continuously on $\rho_\alpha(k, p)$ and $\sigma_\alpha(p_\alpha)$ in our Hölder metric. Therefore, for $\varepsilon \to 0$, $\rho_\alpha(k, p; \varepsilon)$ and $\sigma_\alpha(p_\alpha; \varepsilon)$ are respectively given by $\rho_\alpha(k, p)$ and $\sigma_\alpha(p_\alpha)$ up to $o(1)$. Returning to (7.29), we obtain

$$([\mathbf{R}_0(\lambda + i\varepsilon) - \mathbf{R}_0(\lambda - i\varepsilon)]\hat{\chi}_\alpha, \hat{\chi}_\alpha) + (\mathbf{R}_0(\lambda + i\varepsilon)\sum_{\beta \neq \alpha} \chi_\beta, \chi_\alpha) -$$
$$-\left(\mathbf{R}_0(\lambda - i\varepsilon)\chi_\alpha, \sum_{\beta \neq \alpha} \chi_\beta\right) = o(1),$$

and adding up these equalities for all α, we get

$$\sum_\alpha ([\mathbf{R}_0(\lambda + i\varepsilon) - \mathbf{R}_0(\lambda - i\varepsilon)]\hat{\chi}_\alpha, \hat{\chi}_\alpha) +$$
$$+ \sum_{\substack{\alpha, \beta \\ \alpha \neq \beta}} ([\mathbf{R}_0(\lambda + i\varepsilon) - \mathbf{R}_0(\lambda - i\varepsilon)]\chi_\alpha, \chi_\beta) = o(1). \tag{7.30}$$

Consider now the transition to the limit for $\varepsilon \to 0$ in each of the terms on the left-hand side of (7.30). We have here integrals of the type

$$I_{\alpha\beta} = \int \left\{ \rho_\alpha(k, p) + \frac{\varphi_\alpha(k_\alpha)\sigma_\alpha(p_\alpha)}{\lambda + i\varepsilon + \varkappa_\alpha^2 - \frac{p_\alpha^2}{2n_\alpha}} \right\} \times$$
$$\times \left[\frac{1}{\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda - i\varepsilon} - \frac{1}{\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda + i\varepsilon} \right] \left\{ \overline{\rho_\beta(k, p)} + \frac{\overline{\varphi_\beta(k_\beta)}\,\overline{\sigma_\beta(p_\beta)}}{\lambda - i\varepsilon + \varkappa_\beta^2 - \frac{p_\beta^2}{2n_\beta}} \right\} dkdp.$$

Let us write the singular denominators in a different way. The greatest number of singularities appears in the product of the second terms in the braces. We represent this product in the form

$$2i\varepsilon \frac{1}{\lambda + i\varepsilon + \varkappa_\alpha^2 - \frac{p_\alpha^2}{2n_\alpha}} \frac{1}{\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda - i\varepsilon} \frac{1}{\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda + i\varepsilon} \frac{1}{\lambda - i\varepsilon + \varkappa_\beta^2 - \frac{p_\beta^2}{2n_\beta}} =$$
$$= 2i\varepsilon \frac{1}{\frac{k_\alpha^2}{2m_\alpha} + \varkappa_\alpha^2} \left(\frac{1}{\lambda + i\varepsilon + \varkappa_\alpha^2 - \frac{p_\alpha^2}{2n_\alpha}} + \frac{1}{\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda - i\varepsilon} \right) \times$$
$$\times \left(\frac{1}{\lambda - i\varepsilon + \varkappa_\beta^2 - \frac{p_\beta^2}{2n_\beta}} + \frac{1}{\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda + i\varepsilon} \right) \frac{1}{\frac{k_\beta^2}{2m_\beta} + \varkappa_\beta^2}. \tag{7.31}$$

For $\alpha \neq \beta$ we may take p_α and p_β as the integration variables, and for $\varepsilon \to 0$ the only nonvanishing contribution comes from the product of the singular denominators $\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda - i\varepsilon\right)^{-1}$ and $\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda + i\varepsilon\right)^{-1}$ in (7.31). We obtain

$$I_{\alpha\beta} = \int \left\{ \rho_\alpha(k, p) + \frac{\varphi_\alpha(k_\alpha)\sigma_\alpha(p_\alpha)}{\frac{k_\alpha^2}{2m_\alpha} + \varkappa_\alpha^2} \right\} \frac{2i\varepsilon}{\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda\right)^2 + \varepsilon^2} \times$$
$$\times \left\{ \overline{\rho_\beta(k, p)} + \frac{\overline{\varphi_\beta(k_\beta)\sigma_\beta(p_\beta)}}{\frac{k_\beta^2}{2m_\beta} + \varkappa_\beta^2} \right\} dkdp + o(1) =$$
$$= 2\pi i \int \tilde{\chi}_\alpha(k, p)\overline{\tilde{\chi}_\beta(k, p)}\,\delta\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda\right) dkdp + o(1).$$

In the case $\alpha = \beta$, we take k_α and p_α as integration variables. Then for $\varepsilon \to 0$ the only nonvanishing contribution comes from the product of the denominators $\left(\lambda \mp i\varepsilon + \varkappa_\alpha^2 - \frac{p_\alpha^2}{2n_\alpha}\right)^{-1}$ in (7.31). The result is

$$I_{\alpha\alpha} = 2\pi i \int |\tilde{\chi}_\alpha(k, p)|^2 \delta\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda\right) dkdp +$$

$$+ \int \frac{|\varphi_\alpha(k_\alpha)|^2}{\left(\frac{k_\alpha^2}{2m_\alpha} + \varkappa_\alpha^2\right)^2} |\sigma_\alpha(p_\alpha)|^2 \frac{2i\varepsilon}{\left(\frac{p_\alpha^2}{2n_\alpha} - \varkappa_\alpha^2 - \lambda\right)^2 + \varepsilon^2} dk_\alpha dp_\alpha + o(1).$$

Integration with respect to k_α in the second term gives 1, since

$$\varphi_\alpha(k_\alpha) = \left(\frac{k_\alpha^2}{2m_\alpha} + \varkappa_\alpha^2\right) \psi_\alpha(k_\alpha),$$

where the ψ_α are eigenfunctions of $\mathbf{h}$-type operators, normalized to unity. We finally obtain

$$I_{\alpha\alpha} = 2\pi i \left\{ \int |\tilde{\chi}_\alpha(k, p)|^2 \delta\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda\right) dkdp +\right.$$

$$\left. + \int |\sigma_\alpha(p_\alpha)|^2 \delta\left(-\varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha} - \lambda\right) dp_\alpha \right\} + o(1).$$

We have not distinguished in the above calculations between $\rho_\alpha(k, p)$, $\sigma_\alpha(p_\alpha)$ and $\rho_\alpha(k, p; \varepsilon)$, $\sigma_\alpha(p_\alpha; \varepsilon)$, as this introduces an error of the order $o(1)$ when $\varepsilon \to 0$.

We may now write (7.30) in the form

$$2\pi i \left\{ \int |\tilde{\chi}_{23}(k, p) + \tilde{\chi}_{31}(k, p) + \tilde{\chi}_{12}(k, p)|^2 \delta\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda\right) dkdp +\right.$$

$$\left. + \sum_\alpha \int |\sigma_\alpha(p_\alpha)|^2 \delta\left(-\varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha} - \lambda\right) dp_\alpha \right\} = o(1). \tag{7.32}$$

Since all the terms on the left-hand side of (7.32) are positive and independent of ε, they must vanish individually.

This completes the proof.

Lemma 7.10. Let ω, λ, $\rho_\alpha(k, p)$ *and* $\sigma_\alpha(p_\alpha)$ *be the same as in Lemma 7.9. Then there exists a* $\mu > \frac{1}{2}$, *such that*

$$|\rho_\alpha(k_\alpha + h, p_\alpha) - \rho_\alpha(k_\alpha, p_\alpha)| \leqslant C|h|^\mu; \tag{7.33}$$

$$|\sigma_\alpha(p_\alpha + l) - \sigma_\alpha(p_\alpha)| \leqslant C|l|^\mu; \quad \lambda - \frac{p_\alpha^2}{2n_\alpha} < 0. \tag{7.34}$$

Proof. The estimate (7.33) follows from the fact that $\rho_\alpha(k, p)$ depends on k_α only by way of the kernel $t_\alpha\left(k_\alpha, k'_\alpha, z - \frac{p_\alpha^2}{2n_\alpha}\right)$, which is a sufficiently smooth function of k_α. To obtain the estimate (7.34), consider the expression for $\sigma_\alpha(p_\alpha)$, which follows from (7.17) for $n=2$. In $\sigma_{23}(p_1)$ we encounter integrals of the type

$$\int \frac{\varphi_{23}\left(-p'_2 - \frac{m_2}{m_2+m_3} p_1\right) t_{31}\left(p_1 + \frac{m_1}{m_3+m_1} p'_2, k'_{31}, z - \frac{p'^2_2}{2n_2}\right) \chi_{23}(k', p')}{\left(\frac{1}{2m_{23}}\left(p'_2 + \frac{m_2}{m_2+m_3} p_1\right)^2 + \frac{1}{2n_1} p_1^2 - z\right)\left(\frac{k'^2}{2m} + \frac{p'^2}{2n} - z\right)} dk'dp'$$

$$(z = \lambda + i0).$$

If $\lambda - \frac{p_1^2}{2n_1} < 0$, then the denominator which contains p_1 is nonsingular. The numerator depends on p_1 through sufficiently smooth functions. The proof follows.

Lemma 7.11. Let $\lambda\in\Phi^{(+)}$, *while* $\lambda\neq-\varkappa_\alpha^2$, $\alpha=23, 31, 12$. *Then* λ *is a discrete eigenvalue of* $\mathbf{H}$.

Proof. Let ω be a nontrivial solution of (7.16) with $z=\lambda+i0$, and $\rho_\alpha(k, p)$, $\sigma_\alpha(p_\alpha)$— its components.

Consider the function

$$\psi(k, p; \varepsilon)=\frac{1}{\frac{k^2}{2m}+\frac{p^2}{2n}-\lambda-i\varepsilon}\sum_\alpha\left\{\rho_\alpha(k, p)+\frac{\varphi_\alpha(k_\alpha)\sigma_\alpha(p_\alpha)}{\lambda+i\varepsilon+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}}\right\}. \qquad (7.35)$$

We will show that if $\lambda\neq-\varkappa_\alpha^2$, $\alpha=23, 31, 12$, then $\psi(k, p; \varepsilon)\in\mathfrak{D}$ for any $\varepsilon\geqslant 0$, i. e., $|\psi|$ and $(1+k^2+p^2)|\psi|$ are square-integrable over the entire space, uniformly in ε. The main difficulty arises in the estimation of the integral of $|\psi|^2$ over a finite region. If

$$p_\alpha^2\geqslant 4M(|\lambda|+\varkappa_\alpha^2)+1, \quad \alpha=23, 31, 12,$$

then the denominators in (7.35) are nonsingular and ψ satisfies the estimate

$$|\psi(k, p; \varepsilon)|\leqslant CN(k, p; \theta)(1+k^2+p^2)^{-1}, \qquad (7.36)$$

where θ may be taken $>\frac{1}{2}$, which implies

$$\int_{\Omega_\infty}(1+k^2+p^2)^2|\psi(k, p; \varepsilon)|^2 dkdp<\infty.$$

Here Ω_∞ designates the infinite domain in which the following three conditions are simultaneously fulfilled

$$p_\alpha^2\geqslant 4M(|\lambda|+\varkappa_\alpha^2)+1; \quad k_\alpha^2\geqslant 1; \quad \alpha=23, 31, 12.$$

We represent $\psi(k, p; \varepsilon)$ in the finite domain $E_6-\Omega_\infty$, where E_6 is the entire six-dimensional space, in the form

$$\psi(k, p; \varepsilon)=\frac{1}{\frac{k^2}{2m}+\frac{p^2}{2n}-\lambda-i\varepsilon}\sum_\alpha\left\{\rho_\alpha(k, p)+\frac{\varphi_\alpha(k_\alpha)\sigma_\alpha(p_\alpha)}{\frac{k_\alpha^2}{2m_\alpha}+\varkappa_\alpha^2}\right\}+$$

$$+\sum_\alpha\frac{\varphi_\alpha(k_\alpha)}{\frac{k_\alpha^2}{2m_\alpha}+\varkappa_\alpha^2}\,\frac{\sigma_\alpha(p_\alpha)}{\lambda+i\varepsilon+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}}=\psi_0+\psi_{23}+\psi_{31}+\psi_{12}$$

and estimate each term separately. The dependence of ψ_α on k_α and p_α is factorized, and we only have to prove the square-integrability of the second factor which depends on p_α. This essentially amounts to estimating the integral in the neighborhood of the denominator singularity. This singularity does not arise if $\lambda<-\varkappa_\alpha^2$ and all the functions are then bounded, and hence square-integrable, uniformly in ε. In case $\lambda>-\varkappa_\alpha^2$, we make use of the property (7.23) and subtract from $\sigma_\alpha(p_\alpha)$ in the numerator its vanishing value for $\frac{p_\alpha^2}{2n_\alpha}=\lambda+\varkappa_\alpha^2$; we then have

$$\left|\frac{\sigma_\alpha(p_\alpha)}{\lambda+i\varepsilon+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}}\right|=\frac{\left|\sigma_\alpha(p_\alpha)-\sigma_\alpha\left(\frac{p_\alpha}{|p_\alpha|}\sqrt{2n_\alpha(\lambda+\varkappa_\alpha^2)}\right)\right|}{\left|\lambda+i\varepsilon+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}\right|}\leqslant$$

$$\leqslant C\frac{1}{\left|\lambda+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}\right|^{1-\mu}},$$

where $\mu > \frac{1}{2}$, by Lemma (7.11), since $\lambda - \frac{p_\alpha^2}{2n_\alpha} < 0$ in the neighborhood of the denominator singularity. We deduce from this estimate that

$$\int\limits_{\left|\frac{p_\alpha^2}{2n_\alpha} - \lambda - \varkappa_\alpha^2\right| \leqslant \Delta} \left| \frac{\sigma_\alpha(p_\alpha)}{\lambda + i\varepsilon + \varkappa_\alpha^2 - \frac{p_\alpha^2}{2n_\alpha}} \right|^2 dp_\alpha \leqslant$$

$$\leqslant C \int\limits_{-\Delta}^{\Delta} \frac{dt}{|t^2|^{1-\mu}} \leqslant C|\Delta|^{2\mu-1}. \tag{7.37}$$

Here we have passed to the new variable $\frac{p_\alpha^2}{2n_\alpha} - \lambda - \varkappa_\alpha^2 = t$. All the functions are uniformly bounded in the region $\left|\frac{p_\alpha^2}{2n_\alpha} - \lambda - \varkappa_\alpha^2\right| \geqslant \Delta$. It follows that the function ψ_α is square-integrable, uniformly in ε, over a finite region.

Consider now the term ψ_0, assuming for a start that $\lambda > 0$. The main difficulty arises in the estimation of the integral of ψ_0 over the neighborhood of the surface at which the denominator $\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda - i\varepsilon\right)^{-1}$ is singular. We take for this neighborhood the region Ω_Δ defined by

$$\left| \frac{k^2}{2m} + \frac{p^2}{2n} - \lambda \right| \leqslant \Delta.$$

The following additional conditions define four subregions of Ω_Δ:

$$\Omega_0: \frac{p_1^2}{2n_1} \leqslant \lambda; \quad \frac{p_2^2}{2n_2} \leqslant \lambda; \quad \frac{p_3^2}{2n_3} \leqslant \lambda;$$

$$\Omega_\alpha: \frac{p_\alpha^2}{2n_\alpha} \geqslant \lambda; \quad \alpha = 23,\ 31,\ 12.$$

The subregions Ω_0 and Ω_α generally overlap; they clearly cover the entire region Ω_Δ.

We will now show that the singular denominator $\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda - i\varepsilon\right)^{-1}$ is square-integrable over any subregion Ω_α. Taking k_α and p_α as the integration variables, we obtain

$$\int\limits_{\Omega_\alpha} dkdp \frac{1}{\left|\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda - i\varepsilon\right|^2} \leqslant \int\limits_{\Omega_\alpha} dkdp \frac{1}{\left|\frac{k_\alpha^2}{2m_\alpha} + \frac{p_\alpha^2}{2n_\alpha} - \lambda\right|^2} \leqslant$$

$$\leqslant C \int\limits_{x^2+y^2 \leqslant \Delta} \frac{x^2 y\, dx\, dy}{(x^2+y^2)^2} \leqslant C|\Delta|^{1/2}.$$

We have here made the substitution $\frac{k_\alpha^2}{2m_\alpha} = x^2$, $\frac{p_\alpha^2}{2n_\alpha} - \lambda = y^2$. Noting now that the numerator of the function $\psi_0(k, p; \varepsilon)$ is uniformly bounded, we obtain

$$\int\limits_{\Omega_\alpha} |\psi(k, p; \varepsilon)|^2 dkdp \leqslant C|\Delta|^{1/2}. \tag{7.38}$$

Consider now the integral over the subregion Ω_0. Using the property

(7.22) of the function in the numerator of $\psi_0(k, p; \varepsilon)$, we obtain

$$\int_{\Omega_0} |\psi_0(k, p; \varepsilon)|^2 dkdp \leqslant C \int_{\Omega_0} \frac{1}{\left|\frac{k^2}{2m}+\frac{p^2}{2n}-\lambda\right|^2} \times$$

$$\times \sum_\alpha \left\{ \left| \rho_\alpha(k, p) - \rho_\alpha\left(\sqrt{2m_\alpha\left(\lambda-\frac{p_\alpha^2}{2n_\alpha}\right)}\,\frac{k_\alpha}{|k_\alpha|}, p_\alpha\right)\right|^2 + \right.$$

$$\left. + \frac{|\sigma_\alpha(p_\alpha)|^2}{\left|\frac{k_\alpha^2}{2m_\alpha}+\varkappa_\alpha^2\right|^2} \left| \varphi_\alpha(k_\alpha) - \varphi_\alpha\left(\sqrt{2m_\alpha\left(\lambda-\frac{p_\alpha^2}{2n_\alpha}\right)}\,\frac{k_\alpha}{|k_\alpha|}\right)\right|^2 \right\} dkdp.$$

We have subtracted here the vanishing value of the numerator for $\frac{k^2}{2m}+\frac{p^2}{2n}=\lambda$, i. e., for $\frac{k_\alpha^2}{2m_\alpha}=\lambda-\frac{p_\alpha^2}{2n_\alpha}$, $\alpha=23, 31, 12$.

We estimate each term separately, taking k_α and p_α as the integration variables in the corresponding integrals. The functions in the numerator are smooth functions of k_α (the precise statement of this fact is contained in Lemma 7.10). We therefore have

$$\int_{\Omega_0} |\psi_0(k, p; \varepsilon)|^2 dkdp \leqslant C \int_{x^2+y^2\leqslant\Delta} x^2 dx y dy \frac{|x-y|^{2\mu}}{|x^2-y^2|^2} \leqslant C|\Delta|^\mu, \tag{7.39}$$

where we passed to the new variables $\frac{k_\alpha^2}{2m_\alpha}=x^2$, $\lambda-\frac{p_\alpha^2}{2n_\alpha}=y^2$ in each term. Combining (7.38) and (7.39),

$$\int_{\Omega_\Delta} |\psi_0(k, p; \varepsilon)|^2 dkdp \leqslant C(|\Delta|^{1/2}+|\Delta|^\mu). \tag{7.40}$$

The integral of $|\psi_0(k, p; \varepsilon)|^2$ over the remaining part of the finite region is uniformly bounded in view of the boundedness of all the functions.

Let now $\lambda=0$. We will show that $\left(\frac{k^2}{2m}+\frac{p^2}{2n}+i\varepsilon\right)^{-1}$ is square-integrable in the neighborhood of the origin. We have

$$\int_{\left|\frac{k^2}{2m}+\frac{p^2}{2n}\right|\leqslant\Delta} dkdp \frac{1}{\left|\frac{k^2}{2m}+\frac{p^2}{2n}-i\varepsilon\right|^2} \leqslant C \int_{|x^2+y^2|\leqslant\Delta} x^2 dx y^2 dy \frac{1}{(x^2+y^2)^2} \leqslant C|\Delta|, \tag{7.41}$$

where we have again substituted $\frac{k^2}{2m}=x^2$; $\frac{p^2}{2n}=y^2$.

Finally, when $\lambda<0$ the denominator $\left(\frac{k^2}{2m}+\frac{p^2}{2n}-\lambda-i\varepsilon\right)^{-1}$ is nonsingular, so that $\psi_0(k, p; \varepsilon)$ is square-integrable in this case as well. We have thereby proved that

$$\int_{|E_\varepsilon-\Omega_\infty|} |\psi_0(k, p; \varepsilon)|^2 dkdp \leqslant C.$$

But in the region $E_6-\Omega_\infty$

$$1+p^2+k^2\leqslant C.$$

This completes the proof that $\psi(k, p; \varepsilon)\in\mathfrak{D}$.

We shall now show that $\psi(k, p; 0)$ satisfies the equation

$$\mathbf{H}\psi=\lambda\psi. \tag{7.42}$$

To this end we multiply the relation (7.28) by a smooth finite function $f(k, p)$ and integrate with respect to k and p. For $\varepsilon\to 0$ this gives, up to $o(1)$,

$$(f, \chi_\alpha(\varepsilon)+\mathbf{V}_\alpha\mathbf{R}_0(\lambda+i\varepsilon)\sum_\beta\chi_\beta(\varepsilon))=o(1). \tag{7.43}$$

Here, in contrast to (7.28), we explicitly indicate the dependence of χ_α on ε. Note that

$$\mathbf{R}_0(\lambda+i\varepsilon)\sum_\beta\chi_\beta(\varepsilon)=\psi(\varepsilon).$$

Summing (7.43) over α, we get

$$([\mathbf{H}_0+\mathbf{V}_{23}+\mathbf{V}_{31}+\mathbf{V}_{12}-\lambda]f,\quad \psi(\varepsilon))=o(1), \tag{7.44}$$

where the functions f, $\mathbf{H}_0f$, $\mathbf{V}_\alpha f$, $\alpha=23,\ 31,\ 12$, may be assumed to belong to the class $\mathfrak{M}(\theta,\ \mu)$. We now observe that for any $f(k,\ p)\in\mathfrak{M}(\theta,\ \mu)$

$$(f,\ \psi(\varepsilon))=(f,\ \psi(0))+o(1). \tag{7.45}$$

Indeed, recalling the definition (7.35) of $\psi(k,\ p;\ \varepsilon)$, we find that on the left-hand side of (7.45) there appear several singular integrals, the numerators of whose integrands are continuous in the Hölder sense. We may thus pass to the limit in (7.44) and obtain

$$(f,\ (\mathbf{H}-\lambda\mathbf{E})\psi(0))=0,$$

which implies (7.42) on account of the arbitrariness of f.

Let us now show that $\psi\neq 0$. If $\psi(k,\ p;\ 0)=0$, then (7.43) gives

$$\int f(k,\ p)\chi_\alpha(k,\ p;\ \lambda+i\varepsilon)\,dkdp=o(1)$$

for any smooth function $f(k,\ p)$, and it follows from the equation which relates $\rho_\alpha(k,\ p)$, $\sigma_\alpha(p_\alpha)$ and $\chi_\alpha(k,\ p;\ \lambda+i0)$ that $\rho_\alpha(k,\ p)$ and $\sigma_\alpha(p_\alpha)$ both vanish, i. e., $\omega=0$, which contradicts our assumption.

Thus, we have shown that if $\lambda\in\Phi^{(+)}$ and $\lambda\neq-\varkappa_\alpha^2, \alpha=23, 31, 12$, then equation (7.42) possesses a nontrivial solution $\psi\in\mathfrak{D}$, i. e., λ is a discrete eigenvalue of the operator $\mathbf{H}$.

This completes the proof.

One consequence of Lemma 7.11 is that the set $\Phi^{(+)}$ is denumerable.

Lemma 7.12. Only the points $-\varkappa_\alpha^2$, $\alpha=23,\ 31,\ 12$ *may be limit points of the set* $\Phi^{(+)}$.

Proof. We assume the opposite, and let λ_0, $\lambda_0\neq-\varkappa_\alpha^2$, $\alpha=23,\ 31,\ 12$, be a limit point of $\Phi^{(+)}$. Consider a sequence of points λ_n, and the associated sequence of solutions of (7.16) with $z=\lambda_n+i0$, constructed in the proof of Lemma 7.8, and such that for $n\to\infty$, $\lambda_n\to\lambda_0$, and the ω_n converge strongly to the element ω_0, which is itself a solution of (7.16) for $z=\lambda_0+i0$, and are normalized

$$\|\omega_n\|_{\theta,\mu}=1,\quad n=0,\ 1,\ 2,\ \ldots, \tag{7.46}$$

where θ, μ satisfy $\theta<\bar{\theta}$ and $\mu<\bar{\mu}$. The λ_n may obviously be considered to lie in an interval which does not contain the points $-\varkappa_\alpha^2$, $\alpha=23,\ 31,\ 12$. We construct from the components $\rho_\alpha^{(n)}(k,\ p)$ and $\sigma_\alpha^{(n)}(p_\alpha)$ of the elements ω_n the functions $\psi_n(k,\ p)$, exactly as in the proof of Lemma 7.11. These functions belong, by that lemma, to $\mathfrak{D}$, are eigenfunctions of the operator $\mathbf{H}$ with eigenvalues λ_n, and hence must be orthogonal:

$$(\psi_0,\ \psi_n)=\int\psi_0(k,\ p)\overline{\psi_n(k,\ p)}\,dkdp=0. \tag{7.47}$$

We may pass to the limit in (7.47) for $n\to\infty$. To show this we divide the domain of integration into the infinite region Ω_∞, the neighborhood of the singularities of the denominators Ω_Δ, and the remaining region Ω. By a suitable choice of Ω_∞ the integral over it may be made as small as desired, uniformly in n, since in this domain all the functions $\psi_n(k, p)$ satisfy condition (7.36) for $\theta > \frac{1}{2}$. The integral over Ω_Δ may also be made arbitrarily small by applying estimates of the type (7.37), (7.40) and (7.41). Finally, the integral over Ω approaches a limit when $n \to \infty$, since the functions in the numerator of $\psi_n(k, p)$ converge uniformly, and the denominators are uniformly bounded.

We conclude that $(\psi_0, \psi_0) = 0$, which implies that $\omega = 0$, which contradicts (7.46) for $n = 0$. Thus, the point $\lambda_0 \neq -\varkappa_\alpha^2$, $\alpha = 23, 31, 12$ cannot be a limit point of the set $\Phi^{(+)}$. This completes the proof.

Everything said with regard to the set $\Phi^{(+)}$, beginning with Lemma 7.8, holds equally for the set $\Phi^{(-)}$. Let us denote by Φ the union of $\Phi^{(+)}$ and $\Phi^{(-)}$.

The foregoing results are summed up in the following.

Theorem 7.1. Let the conditions stated at the beginning of § 5 be fulfilled. Then, for all z *in the complex plane* $\Pi_{-\varkappa^2}$, *the operator* $\mathbf{A}(z)$ *and all its powers are defined in* $\mathfrak{B}(\theta, \mu)$, *where* $\theta < \tilde{\theta}$, $\mu < \tilde{\mu}$, *on a dense set consisting of the elements* $\omega \in \mathfrak{B}(\theta, \mu')$, $\mu' > \mu$. *For* $n \geqslant 5$ *the operator* $\mathbf{A}^n(z)$ *may be extended over the entire* $\mathfrak{B}(\theta, \mu)$ *to a completely continuous operator with a smooth dependence on* z. *The set* Φ *of all points* z, *for which the homogeneous equation (7.16) admits a nontrivial solution, lies in a finite interval on the real axis, is denumerable, closed, and may have as its limit points only the points* $-\varkappa_\alpha^2$, $\alpha = 23, 31, 12$. *All the points of* Φ, *except possibly the limit points, belong to the discrete spectrum of the operator* $\mathbf{H}$.

We shall now make use of the following general proposition.

Let the operator $\mathbf{A}$, *defined in the Banach space* $\mathfrak{B}$ *on the dense domain* $\mathfrak{D}(\mathbf{A})$, *be such that (i) all its powers* $\mathbf{A}^n, n = 1, 2, \dots$ *are defined on* $\mathfrak{D}(\mathbf{A})$; *(ii) for* $n \geqslant N$, *where* N *is some fixed number, the* $\mathbf{A}^n$ *may be extended over the entire space* $\mathfrak{B}$ *into completely continuous operators; (iii) the equation*

$$\omega = \mathbf{A}\omega$$

has no nontrivial solutions in $\mathfrak{D}(\mathbf{A})$. *Then, if* $\omega_0 \in \mathfrak{D}(\mathbf{A})$, *the equation*

$$\omega = \omega_0 + \mathbf{A}\omega$$

possesses a unique solution $\omega \in \mathfrak{B}$.

This proposition follows for bounded $\mathbf{A}$ from a known theorem of S. M. Nikol'skii /8/. Our more general proposition may be proved almost exactly as this last theorem, the proof of which may be found, for example, in /9/ or /10/.

Let us denote by $\Pi'_{-\varkappa^2}$ the complex plane $\Pi_{-\varkappa^2}$ punctured by excluding the neighborhoods of the points of Φ. Theorem 7.1 and the above proposition then give

Theorem 7.2. Let the conditions of Theorem 7.1 be fulfilled, and let $\omega_0 \in \mathfrak{B}(\theta, \mu)$. *Then the equation*

$$\omega(z) = \omega_0 + \mathbf{A}(z)\,\omega(z)$$

has for all z *in* $\Pi'_{-\varkappa^2}$ *a unique solution* $\omega(z) \in \mathfrak{B}(\theta_1, \mu_1)$, *where* $\theta_1 < \min(\tilde{\theta}, \theta)$, $\mu_1 < \min(\tilde{\mu}, \mu)$ *and*

$$\|\omega(z)\|_{\theta_1, \mu_1} \leqslant C(|z|),$$

$$\|\omega(z + \Delta) - \omega(z)\|_{\theta_1, \mu_1} \leqslant C(|z|)\,|\Delta|^{\delta}, \quad \delta < \tilde{\mu} - \mu_1.$$

The constants $C(|z|)$ *do not increase faster than a fixed power of* $|z|$ *when* $|z| \to \infty$.

§ 8. Expansion theorem for the eigenfunctions of $\mathbf{h}$

It was assumed in § 4 that the discrete spectrum of the operator $\mathbf{h}$ consists of a single, negative and nondegenerate eigenvalue. In this section we shall describe the continuous spectrum of $\mathbf{h}$.

We split $\mathbf{h}$ into the sum

$$\mathbf{h}=\mathbf{h}_d+\mathbf{h}_c, \tag{8.1}$$

where $\mathbf{h}_d$ is the one-dimensional operator

$$\mathbf{h}_d f(k)=-\varkappa^2(f,\psi)\psi(k), \tag{8.2}$$

which constitutes the invariant part of $\mathbf{h}$ in the subspace spanned by its discrete eigenfunction spectrum. The following theorem, which will be proved in this section, makes a rigorous statement concerning the continuous spectrum.

Theorem 8.1. The operator $\mathbf{h}_c$, *considered in the subspace* $\mathfrak{h}_c$ *which is orthogonal to the eigenelement* $\psi(k)$, *is unitarily equivalent to the operator* $\mathbf{h}_0$, *i.e., there exists an isometric operator* $\mathbf{u}$ *such that the following relations are valid*

$$\mathbf{u}^*\mathbf{u}=\mathbf{e};\quad \mathbf{u}\mathbf{u}^*=\mathbf{e}-\mathbf{p};\quad \mathbf{h}\mathbf{u}=\mathbf{u}\mathbf{h}_0, \tag{8.3}$$

where $\mathbf{p}$ *is the projection operator upon the element* ψ.

Below we shall construct explicitly an operator $\mathbf{u}$ which satisfies relations (8.3), and thereby prove Theorem 8.1.

Let $\mathfrak{d}_0$ be a set of smooth and finite functions $f(k)$ which is dense in $\mathfrak{h}$. The integral

$$g(k;\ \varepsilon_1,\ \varepsilon_2;\ f)=\int\frac{t\left(k,\ k',\ \dfrac{k'^2}{2m}+i\varepsilon_1\right)}{\dfrac{k^2}{2m}-\dfrac{k'^2}{2m}-i\varepsilon_2}f(k')\,dk' \tag{8.4}$$

is meaningful for any $f(k)\in\mathfrak{d}_0$.

Lemma 8.1. Let $f(k)\in\mathfrak{d}_0$. *Then* $g(k;\ \varepsilon_1,\ \varepsilon_2;\ f)$ *satisfies the following estimates*

$$|g(k;\ \varepsilon_1,\ \varepsilon_2;\ f)|\leqslant C(1+|k|)^{-(1+\theta_0)};$$

$$|g(k;\ \varepsilon_1',\ \varepsilon_2';\ f)-g(k;\ \varepsilon_1,\ \varepsilon_2;\ f)|\leqslant C(1+|k|)^{-(1+\theta_0)}\left[|\varepsilon_1'-\varepsilon_1|^{\mu}+|\varepsilon_2'-\varepsilon_2|^{\mu}\right],$$

uniformly in $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_1',\ \varepsilon_2'$, *if both* ε_1 *and* ε_1', *and also* ε_2 *and* ε_2' *are either nonnegative or nonpositive.*

The proof follows by the estimates (4.34) and the lemma on singular integrals.

It follows from Lemma 8.1 that one may consider on $\mathfrak{d}_0$ the operator

$$\mathbf{k}(\varepsilon_1,\ \varepsilon_2)f(k)=g(k;\ \varepsilon_1,\ \varepsilon_2;\ f), \tag{8.5}$$

acting in $\mathfrak{h}$, with $\mathbf{k}(\varepsilon_1,\ \varepsilon_2)$ continuous on $\mathfrak{d}_0$ with respect to ε_1 and ε_2, each varying independently within the interval $[-1,0]$ or $[0,\ 1]$. We now introduce the following operators

$$\mathbf{u}^{(+)}=\mathbf{e}-\mathbf{k}(-0,\ -0);\quad \mathbf{u}^{(-)}=\mathbf{e}-\mathbf{k}(+0,\ +0). \tag{8.6}$$

Lemma 8.2. The operators $\mathbf{u}^{(+)}$ *and* $\mathbf{u}^{(-)}$ *are isometric, i.e.,*

$$\mathbf{u}^{(+)*}\mathbf{u}^{(+)}=\mathbf{e};\quad \mathbf{u}^{(-)*}\mathbf{u}^{(-)}=\mathbf{e}. \tag{8.7}$$

Proof. We apply the identity (2.17) in the following form

$$\frac{t(k,k',z_1)}{z_2-z_1}-\frac{t(k,k',z_2)}{z_2-z_1}=\int\frac{t(k,q,z_1)}{\frac{q^2}{2m}-z_1}\,\frac{t(q,k',z_2)}{\frac{q^2}{2m}-z_2}\,dq. \tag{8.8}$$

Setting here

$$z_1=\frac{k^2}{2m}-i\varepsilon,\quad z_2=\frac{k'^2}{2m}+i\varepsilon,$$

so that

$$z_2-z_1=-\frac{k^2}{2m}+\frac{k'^2}{2m}+2i\varepsilon,$$

we multiply (8.8) by $f(k')\overline{f'(k)}$, where $f\in\mathfrak{d}_0$ and $f'\in\mathfrak{d}_0$, and integrate with respect to k and k'. The integrals converge absolutely for $\varepsilon\neq 0$, so that the order of integration is immaterial. By the symmetry property of the kernel $t(k,k',z)$ (cf. (2.15))

$$t(k,k',z)=\overline{t(k',k,\bar{z})} \tag{8.9}$$

and the definitions (8.4), (8.5), the result may be written in the form

$$(\mathbf{k}(\varepsilon,2\varepsilon)f,f')+(f,\mathbf{k}(\varepsilon,2\varepsilon)f')=(\mathbf{k}(\varepsilon,\varepsilon)f,\mathbf{k}(\varepsilon,\varepsilon)f'). \tag{8.10}$$

Passing to the limit for $\varepsilon\to+0$, or $\varepsilon\to-0$, we obtain in terms of the operators $\mathfrak{u}^{(-)}$ and $\mathfrak{u}^{(+)}$

$$(\mathfrak{u}^{(\pm)}f,\mathfrak{u}^{(\pm)}f')=(f,f'). \tag{8.11}$$

It follows from (8.11) that the operators $\mathfrak{u}^{(\pm)}$ are bounded and may therefore be extended by closure over the entire space $\mathfrak{h}$, while (8.11) remains valid. This completes the proof.

Lemma 8.3. Let $\operatorname{Im} z\neq 0$. *Then*

$$\mathbf{r}(z)\mathfrak{u}^{(\pm)}=\mathfrak{u}^{(\pm)}\mathbf{r}_0(z). \tag{8.12}$$

Proof. We apply again the identity (8.8), setting there $z_1=z$ and $z_2=\frac{k'^2}{2m}+i\varepsilon$, so that

$$z_2-z_1=\frac{k'^2}{2m}-z+i\varepsilon.$$

We multiply both sides of (8.8) by $f(k')\in\mathfrak{d}_0$ and integrate with respect to k', writing the denominator in the second term on the left-hand side of (8.8) as follows

$$\frac{1}{\frac{k'^2}{2m}-z+i\varepsilon}=-\frac{1}{\frac{k^2}{2m}-\frac{k'^2}{2m}-i\varepsilon}+\left(\frac{k^2}{2m}-z\right)\frac{1}{\frac{k^2}{2m}-\frac{k'^2}{2m}-i\varepsilon}\,\frac{1}{\frac{k'^2}{2m}-z+i\varepsilon}.$$

The result may be written, using the notations introduced above, as

$$\mathbf{t}(z)\mathbf{r}_0(z-i\varepsilon)f+\mathbf{k}(\varepsilon,\varepsilon)f-(\mathbf{h}_0-z\mathbf{e})\mathbf{k}(\varepsilon,\varepsilon)\mathbf{r}_0(z-i\varepsilon)f=$$
$$=\mathbf{t}(z)\mathbf{r}_0(z)\mathbf{k}(\varepsilon,\varepsilon)f.$$

Multiplying both sides by $\mathbf{r}_0(z)$, adding to them $\mathbf{r}_0(z)f$ and rearranging terms, we obtain

$$[\mathbf{r}_0(z)-\mathbf{r}_0(z)\mathbf{t}(z)\mathbf{r}_0(z-i\varepsilon)]f-[\mathbf{r}_0(z)-\mathbf{r}_0(z)\mathbf{t}(z)\mathbf{r}_0(z)]\mathbf{k}(\varepsilon,\varepsilon)f=$$
$$=[\mathbf{r}_0(z)-\mathbf{k}(\varepsilon,\varepsilon)\mathbf{r}_0(z-i\varepsilon)]f. \tag{8.13}$$

Passing to the limit for $\varepsilon\to\mp 0$, and making use of the expression (2.10) for $\mathbf{r}(z)$ in terms of $\mathbf{t}(z)$, and of definition (8.6), we write (8.13) as

$$\mathbf{r}(z)\mathfrak{u}^{(\pm)}f = \mathfrak{u}^{(\pm)}\mathbf{r}_0(z)f. \tag{8.14}$$

Since $\mathfrak{d}_0$ is dense in $\mathfrak{h}$ and all the operators in (8.14) are bounded, this equality holds for any $f \in \mathfrak{h}$. This proves the lemma.

Corollary. The following relation holds for any bounded function $\varphi(x)$, $-\infty < x < \infty$

$$\varphi(\mathbf{h})\mathfrak{u}^{(\pm)} = \mathfrak{u}^{(\pm)}\varphi(\mathbf{h}_0). \tag{8.15}$$

Observe that if we formally consider the "kernels" of the operators $\mathfrak{u}^{(\pm)}$ and write (8.15) in terms of these kernels for the case $\varphi(x) = x$, we get

$$\frac{k^2}{2m} u^{(\pm)}(k, k^{(0)}) + \int v(k-k')\, u^{(\pm)}(k', k^{(0)})\, dk' = \frac{k^{(0)^2}}{2m} u^{(\pm)}(k, k^{(0)}).$$

The kernels $u^{(\pm)}(k, k^{(0)})$, which may be regarded as the generalized functions

$$u^{(\pm)}(k, k^{(0)}) = \delta(k - k^{(0)}) - \frac{t\left(k, k^{(0)}, \dfrac{k^{(0)^2}}{2m} \mp i0\right)}{\dfrac{k^2}{2m} - \dfrac{k^{(0)^2}}{2m} \pm i0}, \tag{8.16}$$

are seen to be eigenfunctions of the continuous spectrum of $\mathbf{h}$. Relation (8.7) may be called the orthogonality condition for these functions. In configuration representation, i. e., after a Fourier transformation with respect to the variable k, the functions $u^{(\pm)}(k, k^{(0)})$ become the ordinary functions $\psi^{(\pm)}(x, k^{(0)})$ which satisfy the equation

$$-\frac{1}{2m}\nabla^2\psi^{(\pm)}(x, k^{(0)}) + v(x)\,\psi^{(\pm)}(x, k^{(0)}) = \frac{k^{(0)^2}}{2m}\psi^{(\pm)}(x, k^{(0)})$$

and behave asymptotically for large $|x|$ as

$$\psi^{(\pm)}(x, k^{(0)}) = \exp\{i(x, k^{(0)})\} + f^{(\pm)}\left(\frac{x}{|x|}, k^{(0)}\right)\frac{\exp\{\mp i\,|k^{(0)}|\,|x|\}}{|x|},$$

where

$$f^{(\pm)}\left(\frac{x}{|x|}, k^{(0)}\right) = 2m2\pi^2 t\left(|k^{(0)}|\frac{x}{|x|}, k^{(0)}, \frac{k^{(0)^2}}{2m} \mp i0\right).$$

The operators $\mathfrak{u}^{(\pm)}$ are thus closely related to the solutions of the stationary scattering problem for the operator $\mathbf{h}$.

It is easily verified that the kernels $u^{(\pm)}(k, k^{(0)})$ may also be obtained by another procedure, mentioned in the introduction. Namely, taking the convolution of the kernel of the resolvent $\mathbf{r}(z)$ with an eigenfunction of $\mathbf{h}_0$, i. e., $\delta(k - k^{(0)})$, and calculating the residue at $z = \frac{k^{(0)^2}}{2m}$, we obtain

$$\lim_{\varepsilon\to\pm0} -i\varepsilon\int r\left(k, k', \frac{k^{(0)^2}}{2m} + i\varepsilon\right)\delta(k' - k^{(0)})\,dk' = \lim_{\varepsilon\to\pm0} -i\varepsilon r\left(k, k^{(0)}, \frac{k^{(0)^2}}{2m} + i\varepsilon\right) =$$

$$= \lim_{\varepsilon\to\pm0}\left\{\frac{-i\varepsilon\delta(k - k^{(0)})}{\dfrac{k^2}{2m} - \dfrac{k^{(0)^2}}{2m} - i\varepsilon} - \frac{1}{\dfrac{k^2}{2m} - \dfrac{k^{(0)^2}}{2m} - i\varepsilon}\, t\left(k, k^{(0)}, \frac{k^{(0)^2}}{2m} + i\varepsilon\right)\frac{-i\varepsilon}{-i\varepsilon}\right\} =$$

$$= \delta(k - k^{(0)}) - \frac{t\left(k, k^{(0)}, \dfrac{k^{(0)^2}}{2m} \pm i0\right)}{\dfrac{k^2}{2m} - \dfrac{k^{(0)^2}}{2m} \mp i0},$$

which is again (8.16). Here we have used the following expression for the kernel of $\mathbf{r}(z)$

$$r(k, k', z)=\frac{\delta(k-k')}{\frac{k^2}{2m}-z}-\frac{1}{\frac{k^2}{2m}-z}\,t(k, k', z)\,\frac{1}{\frac{k'^2}{2m}-z},$$

which is simply (2.10) written in terms of the kernels of the corresponding operators. It must be stressed that all this is only heuristic reasoning.

We now go on to prove the completeness of these eigenfunctions. For this we need to relate the operators $\mathbf{u}^{(\pm)}$ with the spectral function of $\mathbf{h}$, which is naturally done by applying the well-known relation between the resolvent and the spectral function of a self-adjoint operator. Denoting the spectral function of $\mathbf{h}$ by $\mathbf{e}(\lambda)$, we have

$$\int\limits_{\mu}^{\nu} d(\mathbf{e}(\lambda)f, f')=\frac{1}{2\pi i}\lim_{\varepsilon\to+0}\int\limits_{\mu}^{\nu}([\mathbf{r}(\lambda+i\varepsilon)-\mathbf{r}(\lambda-i\varepsilon)]f, f')\,d\lambda. \tag{8.17}$$

Let us introduce a convenient representation for the integrand on the right-hand side. Expressing the right-hand side of the Hilbert identity

$$\mathbf{r}(\lambda+i\varepsilon)-\mathbf{r}(\lambda-i\varepsilon)=2i\varepsilon\mathbf{r}(\lambda-i\varepsilon)\mathbf{r}(\lambda+i\varepsilon)$$

in terms of $\mathbf{r}_0(z)$ and $\mathbf{t}(z)$, we obtain

$$\begin{aligned}\mathbf{r}(\lambda+i\varepsilon)-\mathbf{r}(\lambda-i\varepsilon)&=[\mathbf{e}-\mathbf{r}_0(\lambda-i\varepsilon)\mathbf{t}(\lambda-i\varepsilon)]\times\\ &\times[\mathbf{r}_0(\lambda+i\varepsilon)-\mathbf{r}_0(\lambda-i\varepsilon)][\mathbf{e}-\mathbf{t}(\lambda+i\varepsilon)\mathbf{r}_0(\lambda+i\varepsilon)].\end{aligned} \tag{8.18}$$

Lemma 8.4. Let $f\in\mathfrak{d}_0$. *Then*

$$g(k; z; f)=f(k)-\int\frac{t(k, k', z)}{\frac{k'^2}{2m}-z}f(k')\,dk' \tag{8.19}$$

satisfies the estimates

$$\begin{gathered}|g(k; z; f)|\leqslant C(1+|k|)^{-(1-\theta_0)};\\ |g(k+h; z+\Delta; f)-g(k; z; f)|\leqslant\\ \leqslant C(1+|k|)^{-(1+\theta_0)}[|h|^{\mu}+|\Delta|^{\mu}];\quad \mu<\frac{1}{2}\end{gathered}$$

for any $z\in\Pi_0$, *excluding the neighborhood of the point* $z=-\varkappa^2$.

The proof follows from the estimates (4.33), (4.34) and the lemma on singular integrals.

Lemma 8.5. Let $f(k)\in\mathfrak{d}_0$ *and* $f'(k)\in\mathfrak{d}_0$. *Then* $(\mathbf{e}(\lambda)f, f')$ *is continuously differentiable with respect to* λ *for* $\lambda>0$, *and*

$$\begin{gathered}\frac{d}{d\lambda}(\mathbf{e}(\lambda)f, f')=\\ =\int g\left(k; \frac{k^2}{2m}\pm i0; f\right)\overline{g\left(k, \frac{k^2}{2m}\pm i0; f'\right)}\,\delta\left(\frac{k^2}{2m}-\lambda\right)dk.\end{gathered} \tag{8.20}$$

The proof follows by passing to the limit for $\varepsilon\to\pm0$ in the formula

$$\begin{gathered}([\mathbf{r}(\lambda+i\varepsilon)-\mathbf{r}(\lambda-i\varepsilon)]f, f')=\\ =\int g(k; \lambda+i\varepsilon; f)\frac{2i\varepsilon}{\left(\frac{k^2}{2m}-\lambda\right)^2+\varepsilon^2}\overline{g(k, \lambda+i\varepsilon; f')}\,dk,\end{gathered} \tag{8.21}$$

which is obtained directly from (8.18) and (8.19).

It follows from the estimates of Lemma 8.4 for the function $g(k; z; f)$, that for $\varepsilon\to0$ the right-hand side of (8.21) tends to a limit which coincides with the right-hand side of (8.20), which is a continuous function of λ. Differentiating (8.17) with respect to the upper limit and applying the last relations, we obtain (8.20). This completes the proof.

Lemma 8.6. The following relation is valid

$$\int\limits_{-\infty}^{0} d\mathbf{e}(\lambda)=\mathbf{p}. \tag{8.22}$$

Proof. For any $\lambda<0$, $\lambda\neq-\varkappa^2$ we may pass to the limit in (8.21) for $\varepsilon\to+0$, obtaining

$$\frac{d\mathbf{e}(\lambda)}{d\lambda}=0;\quad \lambda<0,\quad \lambda\neq-\varkappa^2. \tag{8.23}$$

The operator $\mathbf{h}$ has a discrete eigenvalue at $\lambda=-\varkappa^2$; consequently,

$$\mathbf{e}(-\varkappa^2+0)-\mathbf{e}(-\varkappa^2-0)=\mathbf{p}. \tag{8.24}$$

The relation (8.22) follows from (8.23) and (8.24).

Lemma 8.7. The following relation is valid

$$\mathbf{u}^{(\pm)}\mathbf{u}^{(\pm)*}=\mathbf{e}-\mathbf{p}. \tag{8.25}$$

Proof. We first show that if $f\in\mathfrak{d}_0$, then

$$g\left(k;\ \frac{k^2}{2m}\pm i0;\ f\right)=\mathbf{u}^{(\pm)*}f(k). \tag{8.26}$$

By definition of $\mathbf{k}(\varepsilon,\varepsilon)$, any $f'(k)\in\mathfrak{d}_0$ satisfies

$$(f,\ [\mathbf{e}-\mathbf{k}(\varepsilon,\ \varepsilon)]f')=(f,\ f')-\int f(k)\overline{\left[\int\frac{t\left(k,\ k',\ \frac{k'^2}{2m}+i\varepsilon\right)}{\frac{k^2}{2m}-\frac{k'^2}{2m}-i\varepsilon}f'(k')dk'\right]}dk=$$

$$=(f,\ f')-\int\overline{f'(k)}\left[\int\frac{t\left(k,k',\ \frac{k^2}{2m}-i\varepsilon\right)}{\frac{k'^2}{2m}-\frac{k^2}{2m}+i\varepsilon}f(k')\,dk'\right]dk=$$

$$=\int g\left(k;\ \frac{k^2}{2m}-i\varepsilon;\ f\right)\overline{f'(k)}\,dk. \tag{8.27}$$

We have changed the order of integration, which is permissible for $\varepsilon\neq0$, and applied (8.9). The required relation (8.26) is obtained from (8.27) by passing to the limit for $\varepsilon\to\pm0$. We now integrate (8.20) over λ from 0 to ∞, which in view of (8.26) and (8.22) gives

$$\int\limits_{0}^{\infty} d(\mathbf{e}(\lambda)f,\ f')=(\mathbf{u}^{(\pm)*}f,\ \mathbf{u}^{(\pm)*}f')=([\mathbf{e}-\mathbf{p}]f,\ f'),$$

confirming (8.25).

The preceding results lead to the following theorem.

Theorem 8.2. Any function $f\in\mathfrak{h}$ has a unique representation as

$$f(k)=c\psi(k)+\mathbf{u}^{(\pm)}f^{(\pm)}(k), \tag{8.28}$$

and for an arbitrary bounded function $\varphi(x)$, $-\infty<x<\infty$

$$\varphi(\mathbf{h})f=\varphi(-\varkappa^2)c\psi+\mathbf{u}^{(\pm)}\varphi(\mathbf{h}_0)f^{(\pm)}, \tag{8.29}$$

where the coefficient c and the functions $f^{(\pm)}$ are defined by

$$c=(f,\ \psi);\quad f^{(\pm)}=\mathbf{u}^{(\pm)*}f \tag{8.30}$$

and

$$\|f\|^2=|c|^2+\|f^{(\pm)}\|^2. \tag{8.31}$$

These formulas constitute a simple and rigorous formulation of the expansion theorem in eigenfunctions of the operator $\mathbf{h}$. Formula (8.31) corresponds to the Parseval relation.

Theorem 8.1 follows from Theorem 8.2. We may take $\mathbf{u}^{(+)}$ or $\mathbf{u}^{(-)}$ for the operator $\mathbf{u}$ in Theorem 8.1. Note that any operator of the form

$$\mathbf{u} = \mathbf{u}^{(+)}\mathbf{m}, \tag{8.32}$$

where $\mathbf{m}$ is unitary and commutes with $\mathbf{h}_0$, also possesses the properties stated in this theorem. The operator $\mathbf{u}^{(-)}$ may be expressed in terms of $\mathbf{u}^{(+)}$ by a relation of the type (8.32), and with a proof of this assertion we conclude the present section.

Consider the operator

$$\mathbf{s} = \mathbf{u}^{(+)^*}\mathbf{u}^{(-)}. \tag{8.33}$$

Lemma 8.8. The operator $\mathbf{s}$ *is unitary and commutes with any bounded function of the operator* $\mathbf{h}_0$. *The following relation is valid*

$$\mathbf{u}^{(-)} = \mathbf{u}^{(+)}\mathbf{s}. \tag{8.34}$$

Proof. Let us first show that

$$\mathbf{p}\mathbf{u}^{(+)} = \mathbf{p}\mathbf{u}^{(-)} = 0. \tag{8.35}$$

The product $\mathbf{u}\mathbf{u}^*\mathbf{u}$ may be written in two ways (we omit for brevity the signs ±), viz.,

$$\mathbf{u}\mathbf{u}^*\mathbf{u} = \mathbf{u}(\mathbf{u}^*\mathbf{u}) = \mathbf{u};$$

$$\mathbf{u}\mathbf{u}^*\mathbf{u} = (\mathbf{u}\mathbf{u}^*)\mathbf{u} = (\mathbf{e} - \mathbf{p})\mathbf{u},$$

which directly implies (8.35). We thus have

$$\mathbf{s}^*\mathbf{s} = \mathbf{u}^{(-)^*}\mathbf{u}^{(+)}\mathbf{u}^{(+)^*}\mathbf{u}^{(-)} = \mathbf{u}^{(-)^*}(\mathbf{e} - \mathbf{p})\mathbf{u}^{(-)} = \mathbf{e};$$

$$\mathbf{s}\mathbf{s}^* = \mathbf{u}^{(+)^*}\mathbf{u}^{(-)}\mathbf{u}^{(-)^*}\mathbf{u}^{(+)} = \mathbf{u}^{(+)^*}(\mathbf{e} - \mathbf{p})\mathbf{u}^{(+)} = \mathbf{e},$$

which proves that $\mathbf{s}$ is unitary.

Let us now apply (8.15) in the form

$$\varphi(\mathbf{h})\mathbf{u}^{(-)} = \mathbf{u}^{(-)}\varphi(\mathbf{h}_0);$$

$$\mathbf{u}^{(+)^*}\varphi(\mathbf{h}) = \varphi(\mathbf{h}_0)\mathbf{u}^{(+)^*}.$$

Multiplying the first of these equalities by $\mathbf{u}^{(+)^*}$ on the left, and the second by $\mathbf{u}^{(-)}$ on the right, we obtain

$$\mathbf{s}\varphi(\mathbf{h}_0) = \varphi(\mathbf{h}_0)\mathbf{s}, \tag{8.36}$$

which is the required commutation relation.

Finally, (8.34) follows from the definition (8.33) of $\mathbf{s}$ and the property (8.35). This completes the proof.

Let us now express the operator $\mathbf{s}$ in terms of the kernel $t(k, k', z)$. We substitute in (8.8)

$$z_1 = \frac{k^2}{2m} + 2i\varepsilon, \quad z_2 = \frac{k'^2}{2m} + i\varepsilon,$$

so that

$$z_2 - z_1 = \frac{k'^2}{2m} - \frac{k^2}{2m} - i\varepsilon,$$

multiply the result by $f(k')f'(k)$ and integrate with respect to k and k'. We obtain

$$(f, \mathbf{k}(-2\varepsilon, -\varepsilon)f') + (\mathbf{k}(\varepsilon, -\varepsilon)f, f') = (\mathbf{k}(\varepsilon, \varepsilon)f, \mathbf{k}(-2\varepsilon, -2\varepsilon)f'),$$

which gives on passing to the limit for $\varepsilon \to +0$

$$(\mathbf{u}^{(-)}f,\ \mathbf{u}^{(+)}f') = (f,\ f') + (\mathbf{k}(+0,\ -0)f,\ f') - (\mathbf{k}(+0,\ +0)f,\ f'),$$

whence

$$\mathbf{s}f = f + [\mathbf{k}(+0,\ -0) - \mathbf{k}(+0,\ +0)]f. \tag{8.37}$$

The definition (8.4), (8.5) of the operator $\mathbf{k}(\varepsilon_1,\ \varepsilon_2)$ implies that (8.37) may be written for $f \in \mathfrak{d}_0$ in the form

$$\mathbf{s}f = f(k) - 2\pi i \int t\left(k,\ k',\ \frac{k'^2}{2m} + i0\right) \delta\left(\frac{k^2}{2m} - \frac{k'^2}{2m}\right) f(k')\,dk'.$$

The kernel of s thus bears a close relationship to the scattering amplitude $f(n,\ k)$.

§ 9. Expansion theorem for the eigenfunctions of $\mathbf{H}$

In this section we shall describe the continuous spectrum of the operator $\mathbf{H}$. We shall show, in analogy with what was done in § 8 for $\mathbf{h}$, that the invariant part of $\mathbf{H}$ in the subspace orthogonal to the eigenfunctions of the discrete spectrum is unitarily equivalent to a certain operator $\hat{\mathbf{H}}$, the spectrum of which may be easily described.

Before we go on to prove rigorous propositions, let us consider some heuristic arguments. As was mentioned in the introduction, neither $\mathbf{H}_0$ nor any of the $\mathbf{H}_\alpha$, $\alpha = 23,\ 31,\ 12$, may serve as $\hat{\mathbf{H}}$, since all these operators have less eigenfunctions than $\mathbf{H}$. Stationary scattering theory stipulates that $\mathbf{H}$ should have eigenfunctions corresponding to those of $\mathbf{H}_0$, viz.,

$$\Phi_0(k,\ p;\ k^{(0)},\ p^{(0)}) = \delta(k - k^{(0)})\,\delta(p - p^{(0)}),$$

plus those of $\mathbf{H}_\alpha$, viz.,

$$\Phi_\alpha(k,\ p;\ p_\alpha^{(0)}) = \delta(p_\alpha - p_\alpha^{(0)})\,\psi_\alpha(k_\alpha), \quad \alpha = 23,\ 31,\ 12.$$

The functions $\Phi_0(k,\ p;\ k^{(0)},\ p^{(0)})$ describe the asymptotic free motion of all three bodies, while in the asymptotic state described by, say, $\Phi_{23}(k,\ p;\ p_1^{(0)})$ the bodies 2 and 3 form a bound pair. The functions $\Phi_{31}(k,\ p;\ p_2^{(0)})$ and $\Phi_{12}(k,\ p;\ p_3^{(0)})$ have a similar meaning.

Given the resolvent $\mathbf{R}(z)$ of $\mathbf{H}$, we can derive suitable eigenfunctions of this operator by means of the procedure which was already described in the introduction and in § 8. Let $\mathscr{R}(k,\ p;\ k',\ p';\ z)$ be the kernel of the resolvent of $\mathbf{H}$. Then the functions

$$\Psi_0^{(\pm)}(k,\ p;\ k^{(0)},\ p^{(0)}) = \lim_{\varepsilon \to \mp 0} -i\varepsilon \mathscr{R}\left(k,\ p;\ k^{(0)},\ p^{(0)};\ \frac{k^{(0)^2}}{2m} + \frac{p^{(0)^2}}{2n} + i\varepsilon\right);$$

$$\Psi_\alpha^{(\pm)}(k,\ p;\ p_\alpha^{(0)}) = \lim_{\varepsilon \to \pm 0} -i\varepsilon \int \mathscr{R}\left(k,\ p;\ k'_\alpha,\ p_\alpha^{(0)};\ -\varkappa_\alpha^2 + \frac{p_\alpha^{(0)^2}}{2n_\alpha} + i\varepsilon\right) \psi_\alpha(k'_\alpha)\,dk'_\alpha$$

should constitute a complete set of eigenfunctions of the continuous spectrum of the operator $\mathbf{H}$. Let us see how these functions are expressed in terms of the kernels $\mathscr{M}_{\alpha\beta}(k,\ p:\ k',\ p';\ z)$.

Combining (3.5) and (3.8), we obtain the relation

$$\mathbf{R}(z) = \mathbf{R}_0(z) - \mathbf{R}_0(z) \sum_{\alpha,\,\beta} \mathbf{M}_{\alpha\beta}(z)\,\mathbf{R}_0(z), \tag{9.1}$$

which is written in terms of the kernels of the involved operators as

$$\mathscr{R}(k,\ p;\ k',\ p';\ z)=\delta(k-k')\delta(p-p')\left(\frac{k^2}{2m}+\frac{p^2}{2n}-z\right)^{-1}-$$
$$-\left(\frac{k^2}{2m}+\frac{p^2}{2n}-z\right)^{-1}\sum_{\alpha,\beta}\mathscr{M}_{\alpha\beta}(k,\ p;\ k',\ p';\ z)\left(\frac{k'^2}{2m}+\frac{p'^2}{2n}-z\right)^{-1}. \tag{9.2}$$

The following expression for the functions $\Psi_0^{(\pm)}(k,\ p;\ k^{(0)},\ p^{(0)})$ follows immediately from (9.2)

$$\Psi_0^{(\pm)}(k, p; k^{(0)}, p^{(0)})=\delta(k-k^{(0)})\delta(p-p^{(0)})-$$
$$-\left(\frac{k^2}{2m}+\frac{p^2}{2n}-\frac{k^{(0)^2}}{2m}-\frac{p^{(0)^2}}{2n}\pm i0\right)^{-1}\times$$
$$\times\sum_{\alpha,\beta}\mathscr{M}_{\alpha\beta}\left(k,\ p;\ k^{(0)},\ p^{(0)};\ \frac{k^{(0)^2}}{2m}+\frac{p^{(0)^2}}{2n}\mp i0\right). \tag{9.3}$$

We multiply (9.2) by $\phi_\alpha(k_\alpha)$, integrate with respect to k_α, and make use of the fact that

$$\mathscr{M}_{\alpha\beta}(k,\ p;\ k',\ p';\ z)=$$
$$=t_\alpha\left(k_\alpha,\ k'_\alpha,\ z-\frac{p_\alpha^2}{2n_\alpha}\right)\delta(p_\alpha-p'_\alpha)\delta_{\alpha\beta}+\mathscr{W}_{\alpha\beta}(k,\ p;\ k',\ p';\ z). \tag{9.4}$$

We may expect, in view of relations of the type (5.25), that

$$\int\mathscr{W}_{\alpha\beta}\left(k,\ p;\ k',\ p':\ -\varkappa_\gamma^2+\frac{p_\gamma^{(0)^2}}{2n_\gamma}+i\varepsilon\right)\frac{\psi_\gamma(k'_\gamma)}{\frac{k'^2}{2m}+\frac{p'^2}{2n}+\varkappa_\gamma^2-\frac{p_\gamma^{(0)^2}}{2n_\gamma}-i\varepsilon}\,dk'_\gamma$$

should have for $\beta=\gamma$ a singularity of type $\frac{1}{\varepsilon}$, which ought not to arise when $\beta\neq\gamma$. Hence it follows that the functions $\Psi_\alpha^{(\pm)}\left(k, p; p_\alpha^{(0)}\right)$ may be represented in the form

$$\Psi_\alpha^{(\pm)}(k,\ p;\ p_\alpha^{(0)})=\psi_\alpha(k_\alpha)\delta(p_\alpha-p_\alpha^{(0)})-$$
$$-\left(\frac{k^2}{2m}+\frac{p^2}{2n}+\varkappa_\alpha^2-\frac{p_\alpha^{(0)^2}}{2n_\alpha}\pm i0\right)^{-1}\sum_\beta\mathscr{K}_{\beta\alpha}\left(k, p; p_\alpha^{(0)}; -\varkappa_\alpha^2+\frac{p_\alpha^{(0)^2}}{2n_\alpha}\mp i0\right). \tag{9.5}$$

Here, unlike § 5, we denote by $\mathscr{K}_{\alpha\beta}(k,\ p;\ p'_\beta;\ z)$ the kernels obtained by a formula of the type (5.23) from the components of the kernels $\mathscr{W}_{\alpha\beta}$, and not of $\widetilde{\mathscr{W}}_{\alpha\beta}$. We recall that these two kernels differ in a few of the first iterations of the system (3.20).

Our aim in the present section is to give a rigorous meaning to expressions of the type (9.3) and (9.5), and to prove the possibility of expanding any function into an integral of the functions $\Psi_0^{(\pm)}(k,\ p;\ k^{(0)},\ p^{(0)})$ and $\Psi_\alpha^{(\pm)}(k,\ p;\ p_\alpha^{(0)})$. Exactly as in § 8, we shall not deal with the functions Ψ_0 and Ψ_α themselves, but assign to them isometric operators which diagonalize $\mathbf{H}$.

We start the rigorous treatment with the description of the operator $\hat{\mathbf{H}}$. Consider the four Hilbert spaces $\mathfrak{H}_0$, $\mathfrak{H}_{23}$, $\mathfrak{H}_{31}$, $\mathfrak{H}_{12}$. The elements of $\mathfrak{H}_0$ are the square-integrable functions $f(k,p)$ of two variables, so that this space is identical with $\mathfrak{H}$. The elements of the spaces $\mathfrak{H}_{23}$, $\mathfrak{H}_{31}$, $\mathfrak{H}_{12}$ are the square-integrable functions $f(p)$ of one variable. To distinguish between $\mathfrak{H}_0$ and $\mathfrak{H}$, and also between the individual spaces $\mathfrak{H}_\alpha$, $\alpha=23,\ 31,\ 12$, we similarly label their elements and scalar products; thus $f_0(k,\ p)\in\mathfrak{H}_0$, $f_\alpha(p_\alpha)\in\mathfrak{H}_\alpha$, while $(\cdot,\ \cdot)_0$ and $(\cdot,\ \cdot)_\alpha$ are the scalar products in $\mathfrak{H}_0$ and $\mathfrak{H}_\alpha$.

We denote by $\hat{\mathfrak{H}}$ the direct sum

$$\hat{\mathfrak{H}}=\mathfrak{H}_0\oplus\mathfrak{H}_{23}\oplus\mathfrak{H}_{31}\oplus\mathfrak{H}_{12}. \tag{9.6}$$

The operator $\hat{\mathbf{H}}$ operates in $\hat{\mathfrak{H}}$, and is reduced by the subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_\alpha$, $\alpha = 23, 31, 12$, so that

$$\hat{\mathbf{H}} = \tilde{\mathbf{H}}_0 \oplus \tilde{\mathbf{H}}_{23} \oplus \tilde{\mathbf{H}}_{31} \oplus \tilde{\mathbf{H}}_{12}, \tag{9.7}$$

where $\tilde{\mathbf{H}}_0$ operates in $\mathfrak{H}_0$, and $\tilde{\mathbf{H}}_\alpha$ in $\mathfrak{H}_\alpha$. These operators are defined in their natural domains by

$$\tilde{\mathbf{H}}_0 f_0(k, p) = \left(\frac{k^2}{2m} + \frac{p^2}{2n}\right) f_0(k, p); \tag{9.8}$$

$$\tilde{\mathbf{H}}_\alpha f_\alpha(p_\alpha) = \left(-\varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha}\right) f_\alpha(p_\alpha), \quad \alpha = 23,\ 31,\ 12. \tag{9.9}$$

Let $\mathbf{P}_d$ be the projection operator in $\mathfrak{H}$ on the subspace $\mathfrak{H}_d$, spanned by the eigenfunctions of the discrete spectrum of $\mathbf{H}$, and let $\mathfrak{H}_c$ be the orthogonal complement of $\mathfrak{H}_d$ in $\mathfrak{H}$. The subspace $\mathfrak{H}_c$ reduces $\mathbf{H}$. Let $\mathbf{H}_c$ be the corresponding invariant part of $\mathbf{H}$ in $\mathfrak{H}_c$, and $\hat{\mathbf{E}}$, $\mathbf{E}_0$ and $\mathbf{E}_\alpha$ the unit operators in $\hat{\mathfrak{H}}$, $\mathfrak{H}_0$ and $\mathfrak{H}_\alpha$, respectively.

We will prove the following proposition.

Theorem 9.1. The operators $\mathbf{H}_c$ *and* $\hat{\mathbf{H}}$ *are unitarily equivalent, i.e., there exists an isometric operator* $\mathbf{U}$ *mapping* $\hat{\mathfrak{H}}$ *into* $\mathfrak{H}$, *such that the following relations hold*

$$\mathbf{U}^*\mathbf{U} = \hat{\mathbf{E}}, \quad \mathbf{U}\mathbf{U}^* = \mathbf{E} - \mathbf{P}_d; \quad \mathbf{H}\mathbf{U} = \mathbf{U}\hat{\mathbf{H}}. \tag{9.10}$$

To prove this theorem, we construct explicitly an operator $\mathbf{U}$ which possesses the properties listed in Theorem 9.1. This will involve the kernels $\mathscr{M}_{\alpha\beta}(k, p; k', p'; z)$ of the operators $\mathbf{M}_{\alpha\beta}(z)$ and the associated kernels $\mathscr{W}_{\alpha\beta}(k, p; k', p'; z)$. These last kernels are of the type $\mathfrak{Q}_{\alpha\beta}$. In this section, unlike § 5, we denote by $\mathscr{F}_{\alpha\beta}$, $\mathscr{G}_{\alpha\beta}$, $\tilde{\mathscr{G}}_{\alpha\beta}$ and $\mathscr{H}_{\alpha\beta}$ the components of the kernels $\mathscr{W}_{\alpha\beta}$, and by $\mathscr{K}_{\alpha\beta}$ and $\tilde{\mathscr{K}}_{\alpha\beta}$ the kernels obtained from these components by the formulas (5.23) and (5.24).

We now introduce the notations

$$\mathscr{L}_{\alpha\beta}(k, p; p'_\beta; z) = \varphi_\alpha(k_\alpha)\delta(p_\alpha - p'_\alpha)\delta_{\alpha\beta} + \mathscr{K}_{\alpha\beta}(k, p; p'_\beta; z); \tag{9.11}$$

$$\tilde{\mathscr{L}}_{\alpha\beta}(p_\alpha; k', p'; z) = \tilde{\mathscr{K}}_{\alpha\beta}(p_\alpha; k', p'; z) + \delta_{\alpha\beta}\delta(p_\alpha - p'_\alpha)\overline{\varphi_\alpha(k_\alpha)}. \tag{9.12}$$

By repeating the proof of Lemmas 5.1 and 5.2, we may demonstrate the following relations

$$\int \mathscr{M}_{\alpha\beta}(k, p; k', p'; z)\frac{\psi_\beta(k'_\beta)}{\frac{k'^2}{2m} + \frac{p'^2}{2n} - z}\,dk'_\beta = \mathscr{L}_{\alpha\beta}(k, p; p'_\beta; z)\frac{1}{z + \varkappa_\beta^2 - \frac{p'^2_\beta}{2n_\beta}}; \tag{9.13}$$

$$\int \frac{\overline{\psi_\alpha(k_\alpha)}}{\frac{k^2}{2m} + \frac{p^2}{2n} - z}\,\mathscr{M}_{\alpha\beta}(k, p; k', p'; z)\,dk_\alpha = -\frac{1}{z + \varkappa_\alpha^2 - \frac{p_\alpha^2}{2n_\alpha}}\tilde{\mathscr{L}}_{\alpha\beta}(p_\alpha; k', p'; z). \tag{9.14}$$

The discussion at the beginning of this section shows that it is advisable to consider the expressions

$$\int \frac{\mathscr{M}_{\alpha\beta}\left(k, p; k', p'; \frac{k'^2}{2m} + \frac{p'^2}{2n} + i\varepsilon_1\right)}{\frac{k^2}{2m} + \frac{p^2}{2n} - \frac{k'^2}{2m} - \frac{p'^2}{2n} - i\varepsilon_2}\,f_0(k', p')\,dk'dp'; \tag{9.15}$$

$$\int \frac{\mathscr{L}_{\alpha\beta}\left(k,\ p;\ p_\beta;\ -\varkappa_\beta^2+\frac{p_\beta'^2}{2n_\beta}+i\varepsilon_1\right)}{\frac{k^2}{2m}+\frac{p^2}{2n}+\varkappa_\beta^2-\frac{p_\beta'^2}{2n_\beta}-i\varepsilon_2} f_\beta(p_\beta')\,dp_\beta', \tag{9.16}$$

where ε_1 and ε_2 are real parameters (cf. (9.3), (9.5)). If $\varepsilon_1 \neq 0$ and $\varepsilon_2 \neq 0$, then (9.15) and (9.16) define operators that map $\mathfrak{H}_0$ into $\mathfrak{H}$ and $\mathfrak{H}_\beta$ into $\mathfrak{H}$, respectively. We denote these operators by $\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon_1,\varepsilon_2)$ and $\hat{\mathbf{L}}_{\alpha\beta}(\varepsilon_1,\ \varepsilon_2)$.

Let $\mathfrak{D}_0$ and $\mathfrak{D}_\alpha$ be the dense sets in $\mathfrak{H}_0$ and $\mathfrak{H}_\alpha$ which consist of all smooth, finite functions $f_0(k,\ p)$ and $f_\alpha(p_\alpha)$ that vanish at the surfaces $\frac{k^2}{2m}+\frac{p^2}{2n}=\lambda_n$ and $-\varkappa_\alpha^2+\frac{p_\alpha^2}{2n_\alpha}=\lambda_n$, respectively. Here λ_n are the singular points of the operators $\mathbf{A}(z)$ (cf. § 7). Unlike the case considered in § 8, it has not been possible to estimate the integrals (9.15) and (9.16) uniformly in ε_1 and ε_2 over intervals including their zero values, because of secondary singularities of the first iterations of (3.20) which appear in the kernels $\mathscr{M}_{\alpha\beta}$ (cf. § 6). We therefore cannot prove directly that the operators $\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon_1,\ \varepsilon_2)$ and $\hat{\mathbf{L}}_{\alpha\beta}(\varepsilon_1,\ \varepsilon_2)$ converge strongly on the respective dense sets for $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, and the proof of this fact requires the following indirect approach.

Let $f(k,\ p) \in \mathfrak{M}(\theta,\ \mu)$ with $\theta > \frac{1}{2}$. The class $\mathfrak{M}(\theta,\ \mu)$ was defined in § 5, and the functions $f(k,\ p) \in \mathfrak{M}(\theta,\ \mu)$ constitute a dense set in $\mathfrak{H}$ if $\theta > \frac{1}{2}$. Consider the integrals

$$m_{\alpha\beta}(k,\ p;\ z_1,\ z_2;\ f) = \int \mathscr{M}_{\alpha\beta}(k,\ p;\ k',\ p';\ z_1)\frac{f(k',\ p')}{\frac{k'^2}{2m}+\frac{p'^2}{2n}-z_2}\,dk'dp'. \tag{9.17}$$

It follows from (9.4) and a formula of the type (5.3) for the kernels $\mathscr{W}_{\alpha\beta}$, that the functions $m_{\alpha\beta}(k,\ p;\ z_1,\ z_2;\ f)$ may be represented as

$$m_{\alpha\beta}(k,\ p;\ z_1,\ z_2;\ f) = n_{\alpha\beta}(k,\ p;\ z_1,\ z_2;\ f) + \frac{\varphi_\alpha(k_\alpha)\, l_{\alpha\beta}(p_\alpha;\ z_1,\ z_2;\ f)}{z_1+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}}, \tag{9.18}$$

where

$$l_{\alpha\beta}(p_\alpha;\ z_1,\ z_2;\ f) = \int \mathscr{L}_{\alpha\beta}(p_\alpha;\ k',\ p';\ z_1)\frac{f(k',\ p')}{\frac{k'^2}{2m}+\frac{p'^2}{2n}-z_2}\,dk'dp'. \tag{9.19}$$

Lemma 9.1. *Let $f(k,\ p) \in \mathfrak{M}(\theta,\ \mu')$. Then $n_{\alpha\beta}(k,\ p;\ z_1,\ z_2;\ f)$ and $l_{\alpha\beta}(p_\alpha;\ z_1,\ z_2;\ f)$ respectively, belong to the classes $\mathfrak{M}(\theta,\ \mu)$ and $\mathfrak{N}(\theta,\ \mu)$, $\mu < \mu'$, as functions of k, p and p_α, and satisfy estimates of the type (5.6)–(5.9), uniformly in $z_2 \in \Pi_0$ and z_1 lying in a finite region of the plane $\Pi_{-\varkappa}$, which contains none of the singular points λ_n.*

This might be proved by estimating separately the integrals of all the contributions to the kernel $\mathscr{M}_{\alpha\beta}(k,\ p;\ k',\ p';\ z)$. We will proceed by another way. Consider the integral

$$m_\alpha^{(0)}(k,\ p;\ z_1,\ z_2;\ f) = \int t_\alpha\left(k_\alpha,\ k_\alpha',\ z_1-\frac{p_\alpha^2}{2n_\alpha}\right)\frac{\delta(p_\alpha-p_\alpha')}{\frac{k'^2}{2m}+\frac{p'^2}{2n}-z_2}\, f(k',\ p')\,dk'dp'.$$

By (4.50), $m_\alpha^{(0)}(k, p; z_1, z_2; f)$ may be represented in the form

$$m_\alpha^{(0)}(k, p; z_1, z_2; f) = n_\alpha^{(0)}(k, p; z_1, z_2; f) + \frac{\varphi_\alpha(k_\alpha)\, l_\alpha^{(0)}(p_\alpha; z_2; f)}{z_1 + \varkappa_\alpha^2 - \frac{p_\alpha^2}{2n_\alpha}},$$

where

$$n_\alpha^{(0)}(k, p; z_1, z_2; f) = \int \hat{t}_\alpha \left(k_\alpha, k_\alpha', z_1 - \frac{p_\alpha^2}{2n_\alpha}\right) \frac{\delta(p_\alpha - p_\alpha')}{\frac{k'^2}{2m} + \frac{p'^2}{2n} - z_2} f(k', p')\, dk'dp',$$

$$l_\alpha^{(0)}(p_\alpha; z_2; f) = \int \overline{\varphi_\alpha(k_\alpha')} \frac{\delta(p_\alpha - p_\alpha')}{\frac{k'^2}{2m} + \frac{p'^2}{2n} - z_2} f(k', p')\, dk'dp'.$$

Integrals of this type were considered in § 7. The estimates of Lemma 7.1 show that we may regard the aggregate of functions $n_\alpha^{(0)}(k, p; z_1, z_2; f)\delta_{\alpha\beta}$ and $l_\alpha^{(0)}(p_\alpha; z_2; f)\delta_{\alpha\beta}$ as the components of the element $\omega_\beta^{(0)}(z_1, z_2; f) \in \mathfrak{B}(\theta, \mu)$, which satisfies the estimates

$$\|\omega_\beta^{(0)}(z_1, z_2; f)\|_{\theta, \mu} \leqslant C;$$

$$\|\omega_\beta^{(0)}(z_1 + \Delta_1, z_2 + \Delta_2; f) - \omega_\beta^{(0)}(z_1, z_2; f)\|_{\theta, \mu} \leqslant C[|\Delta_1|^\delta + |\Delta_2|^\delta]$$

with certain indices θ, $\mu < \mu'$ and $\delta < \mu' - \mu$.

It is easily verified that those components of the elements $\omega_\beta(z_1, z_2; f)$ that satisfy the equations

$$\omega_\beta(z_1, z_2; f) = \omega_\beta^{(0)}(z_1, z_2; f) - \mathbf{A}(z_1)\, \omega_\beta(z_1, z_2; f),$$

are the functions in question $n_{\alpha\beta}(k, p; z_1, z_2; f)$ and $l_{\alpha\beta}(p_\alpha; z_1, z_2; f)$. To see this we only have to apply the system of equations

$$\mathcal{M}_{\alpha\beta}(k, p; k', p'; z) = t_\alpha\left(k_\alpha, k_\alpha', z - \frac{p_\alpha^2}{2n_\alpha}\right) \delta(p_\alpha - p_\alpha')\delta_{\alpha\beta} -$$

$$- \int t_\alpha\left(k_\alpha, k_\alpha'', z - \frac{p_\alpha^2}{2n_\alpha}\right) \frac{\delta(p_\alpha - p_\alpha'')}{\frac{k''^2}{2m} + \frac{p''^2}{2n} - z} \sum_{\gamma \neq \alpha} \mathcal{M}_{\gamma\beta}(k'', p''; k', p'; z)\, dk''dp''$$

for the kernels $\mathcal{M}_{\alpha\beta}(k, p; k', p'; z)$, set $z = z_1$, multiply by $\left(\frac{k'^2}{2m} + \frac{p'^2}{2n} - z_2\right)^{-1} f(k', p')$ and integrate with respect to k' and p', keeping in mind the definition of the operator $\mathbf{A}(z)$. Now by Theorem 7.2

$$\|\omega_\beta(z_1, z_2; f)\|_{\theta, \mu} \leqslant C; \quad \theta < \theta_0; \quad \mu < \mu';$$

$$\|\omega_\beta(z_1 + \Delta_1, z_2 + \Delta_2; f) - \omega_\beta(z_1, z_2; f)\|_{\theta, \mu} \leqslant C[|\Delta_1|^\delta + |\Delta_2|^\delta],$$

for all z_1 and z_2 allowed in the formulation of Lemma 9.1. This completes its proof.

We may readily verify with the help of Lemma 9.1 that expressions of the type

$$(\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon_1, \varepsilon_2) f_0, f) = \tilde{m}_{\alpha\beta}(\varepsilon_1, \varepsilon_2; f_0, f); \tag{9.20}$$

$$(\hat{\mathbf{L}}_{\alpha\beta}(\varepsilon_1, \varepsilon_2) f_\beta, f) = \tilde{l}_{\alpha\beta}(\varepsilon_1, \varepsilon_2; f_\beta, f), \tag{9.21}$$

approach a limit for $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$. In fact, we have

Lemma 9.2. Let $f \in \mathfrak{M}(\theta, \mu)$, $f_0 \in \mathfrak{D}_0$ *and* $f_\beta \in \mathfrak{D}_\beta$. *Then the functions* $\tilde{m}_{\alpha\beta}(\varepsilon_1, \varepsilon_2; f_0, f)$ *and* $\tilde{l}_{\alpha\beta}(\varepsilon_1, \varepsilon_2; f_\beta, f)$ *are Hölder functions of* ε_1 *and* ε_2, *when both these parameters vary in the interval [-1, 0] or [0, 1].*

Proof. By definition of the operator $\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon_1, \varepsilon_2)$, we have

$$\tilde{m}_{\alpha\beta}(\varepsilon_1,\ \varepsilon_2;\ f_0,\ f) = (\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon_1,\ \varepsilon_2) f_0,\ f) =$$

$$= \int \overline{f(k,\ p)} \frac{\mathcal{M}_{\alpha\beta}\left(k,\ p;\ k',\ p';\ \frac{k'^2}{2m} + \frac{p'^2}{2n} + i\varepsilon_1\right)}{\frac{k^2}{2m} + \frac{p^2}{2n} - \frac{k'^2}{2m} - \frac{p'^2}{2n} - i\varepsilon_2} f_0(k',\ p')\, dk dp dk' dp' =$$

$$= \int f_0(k,\ p)\, \overline{m_{\beta\alpha}\left(k,\ p;\ \frac{k^2}{2m} + \frac{p^2}{2n} - i\varepsilon_1,\ \ \frac{k^2}{2m} + \frac{p^2}{2n} - i\varepsilon_2;\ f\right)} dk dp. \tag{9.22}$$

Here we have changed the order of integration, which we are allowed to do on account of the absolute convergence of all the integrals for $\varepsilon_1 \neq 0$, $\varepsilon_2 \neq 0$, and made use of the symmetry relation for the kernels $\mathcal{M}_{\alpha\beta}$

$$\mathcal{M}_{\alpha\beta}(k,\ p;\ k',\ p';\ z) = \overline{\mathcal{M}_{\beta\alpha}(k',\ p';\ k,\ p;\ \bar{z})} \tag{9.23}$$

and the definition (9.17) of $m_{\alpha\beta}(k,\ p;\ z_1,\ z_2;\ f)$.

We similarly obtain

$$\tilde{l}_{\alpha\beta}(\varepsilon_1,\ \varepsilon_2;\ f_\beta,\ f) = (\hat{\mathbf{L}}_{\alpha\beta}(\varepsilon_1,\ \varepsilon_2) f_\beta,\ f) =$$

$$= \int f_\beta(p_\beta)\, \overline{l_{\beta\alpha}\left(p_\beta;\ -\varkappa_\beta^2 + \frac{p_\beta^2}{2n_\beta} - i\varepsilon_1;\ -\varkappa_\beta^2 + \frac{p_\beta^2}{2n_\beta} - i\varepsilon_2;\ f\right)}. \tag{9.24}$$

The proof of the lemma follows now from the estimates of Lemma 9.1 for $n_{\alpha\beta}(k,\ p;\ z_1,\ z_2;\ f)$ and $l_{\alpha\beta}(p_\alpha;\ z_1,\ z_2;\ f)$, and from (9.18).

We denote by $\hat{\mathbf{E}}_0$ the identity operator, mapping $\mathfrak{H}_0$ into $\mathfrak{H}$,

$$\hat{\mathbf{E}}_0 f_0 = f: \qquad f_0(k,\ p) = f(k,\ p), \tag{9.25}$$

and define

$$\mathbf{U}_0(\varepsilon) = \hat{\mathbf{E}}_0 + \sum_{\alpha,\beta} \hat{\mathbf{M}}_{\alpha\beta}(\varepsilon,\ \varepsilon). \tag{9.26}$$

Lemma 9.3. The operator $\mathbf{U}_0(\varepsilon)$ *converges strongly for* $\varepsilon \to \pm 0$ *on any element* $f_0 \in \mathfrak{D}_0$. *The limit operators*

$$\mathbf{U}_0^{(\pm)} = \lim_{\varepsilon \to \mp 0} \mathbf{U}_0(\varepsilon) \tag{9.27}$$

may be extended by closure into isometric operators over the entire $\mathfrak{H}_0$, *so that*

$$\mathbf{U}_0^{(\pm)*} \mathbf{U}_0^{(\pm)} = \mathbf{E}_0. \tag{9.28}$$

Proof. We apply Hilbert's identity to the resolvent $\mathbf{R}(z)$, expressed in terms of the kernels $\mathcal{M}_{\alpha\beta}$ (cf. (3.36)),

$$\frac{\mathcal{M}_{\alpha\beta}(k,\ p;\ k',\ p';\ z_1)}{z_2 - z_1} - \frac{\mathcal{M}_{\alpha\beta}(k,\ p;\ k',\ p';\ z_2)}{z_2 - z_1} =$$

$$= \int \sum_\gamma \mathcal{M}_{\alpha\gamma}(k,\ p;\ k'',\ p'';\ z_1) \frac{1}{\frac{k''^2}{2m} + \frac{p''^2}{2n} - z_1} \frac{1}{\frac{k''^2}{2m} + \frac{p''^2}{2n} - z_2} \times$$

$$\times \sum_\gamma \mathcal{M}_{\gamma\beta}(k'',\ p'';\ k',\ p';\ z_2)\, dk'' dp''. \tag{9.29}$$

We now set

$$z_1 = \frac{k^2}{2m} + \frac{p^2}{2n} - i\varepsilon_1, \qquad z_2 = \frac{k'^2}{2m} + \frac{p'^2}{2n} + i\varepsilon_2,$$

so that

$$z_2 - z_1 = -\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \frac{k'^2}{2m} - \frac{p'^2}{2n} - i(\varepsilon_1 + \varepsilon_2)\right),$$

multiply by $f_0(k', p')\overline{f_0'(k, p)}$, where $f_0(k, p)\in\mathfrak{D}_0$ and $f_0'(k, p)\in\mathfrak{D}_0$, and integrate with respect to k, p and k', p'. The result may be written in terms of the operators $\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon_1, \varepsilon_2)$ as follows

$$\begin{aligned}(\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon_2, \varepsilon_1+\varepsilon_2)f_0, \hat{\mathbf{E}}_0 f_0')+(\hat{\mathbf{E}}_0 f_0, \hat{\mathbf{M}}_{\alpha\beta}(\varepsilon_1, \varepsilon_1+\varepsilon_2)f_0')=\\ =(\sum_\gamma \hat{\mathbf{M}}_{\gamma\beta}(\varepsilon_2, \varepsilon_2)f_0, \sum_\gamma \hat{\mathbf{M}}_{\gamma\alpha}(\varepsilon_1, \varepsilon_1)f_0').\end{aligned} \tag{9.30}$$

Here we have changed the order of integration, and applied the symmetry relation (9.23) for the kernels $\mathcal{M}_{\alpha\beta}$.

Upon summation over α and β, (9.30) becomes in terms of the operators $\mathbf{U}_0(\varepsilon_1)$ and $\mathbf{U}_0(\varepsilon_2)$

$$\begin{aligned}(\mathbf{U}_0(\varepsilon_2)f_0, \mathbf{U}_0(\varepsilon_1)f_0')-(f_0, f_0')_0=\\ =\Big(\sum_{\alpha,\beta}\{\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon_2, \varepsilon_1+\varepsilon_2)-\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon_2, \varepsilon_2)\}f_0, \hat{\mathbf{E}}_0 f_0'\Big)+\\ +\Big(\hat{\mathbf{E}}_0 f_0, \sum_{\alpha,\beta}\{\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon_1, \varepsilon_1+\varepsilon_2)-\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon_1, \varepsilon_1)\}f_0'\Big),\end{aligned} \tag{9.31}$$

or, expressing the right-hand side of (9.31) in terms of $m_{\alpha\beta}(\varepsilon_1, \varepsilon_2, f_0, f)$

$$\begin{aligned}(\mathbf{U}_0(\varepsilon_2)f_0, \mathbf{U}_0(\varepsilon_1)f_0')-(f_0, f_0')_0=\\ =\sum_{\alpha,\beta}\{\tilde{m}_{\alpha\beta}(\varepsilon_2, \varepsilon_1+\varepsilon_2; f_0, \hat{\mathbf{E}}_0 f_0')-\tilde{m}_{\alpha\beta}(\varepsilon_2, \varepsilon_2; f_0, \hat{\mathbf{E}}_0 f_0')\}+\\ +\sum_{\alpha,\beta}\{\overline{\tilde{m}_{\alpha\beta}(\varepsilon_1, \varepsilon_1+\varepsilon_2; f_0', \hat{\mathbf{E}}_0 f_0)}-\overline{\tilde{m}_{\alpha\beta}(\varepsilon_1, \varepsilon_1; f_0', \hat{\mathbf{E}}_0 f_0)}\}.\end{aligned} \tag{9.32}$$

By Lemma 9.2 the right-hand side of this equality vanishes when both ε_1 and ε_2 tend to zero from the same side. More accurately, there exists an index $\delta>0$, such that for any f_0 and f_0' in $\mathfrak{D}_0$

$$(\mathbf{U}_0(\varepsilon_2)f_0, \mathbf{U}_0(\varepsilon_1)f_0')-(f_0, f_0')_0=O(|\varepsilon_1|^\delta)+O(|\varepsilon_2|^\delta). \tag{9.33}$$

Let now $f_0\in\mathfrak{D}_0$ and $\frac{\varepsilon_1}{\varepsilon_2}>0$. We have

$$\begin{aligned}d^2\equiv\|[\mathbf{U}_0(\varepsilon_1)-\mathbf{U}_0(\varepsilon_2)]f_0\|^2=(\mathbf{U}_0(\varepsilon_1)f_0, \mathbf{U}_0(\varepsilon_1)f_0)-\\ -2\mathrm{Re}(\mathbf{U}_0(\varepsilon_1)f_0, \mathbf{U}_0(\varepsilon_2)f_0)+(\mathbf{U}_0(\varepsilon_2)f_0, \mathbf{U}_0(\varepsilon_2)f_0).\end{aligned}$$

It follows from (9.33) that the right-hand side is of the order $O(|\varepsilon_1|^\delta)+$ $+O(|\varepsilon_2|^\delta)$, so that d may be made arbitrarily small for sufficiently small ε_1 and ε_2, provided they both have the same sign. This entails the convergence of $\mathbf{U}_0(\varepsilon)$ on $\mathfrak{D}_0$ for $\varepsilon\to 0$.

Let us set in (9.33) $\varepsilon_1=\varepsilon_2=\varepsilon$, and pass to the limit for $\varepsilon\to 0$. We obtain for any $f_0(k, p)$ and $f_0'(k, p)$ in $\mathfrak{D}_0$

$$(\mathbf{U}_0^{(\pm)}f_0, \mathbf{U}_0^{(\pm)}f_0')=(f_0, f_0')_0.$$

In view of the denseness of $\mathfrak{D}_0$ in $\mathfrak{H}_0$, it follows that the operators $\mathbf{U}_0^{(+)}$ and $\mathbf{U}_0^{(-)}$ may be extended by closure into isometric operators on the entire $\mathfrak{H}_0$. This completes the proof.

Let us denote by $\mathbf{U}_\alpha(\varepsilon)$ the following operator that maps $\mathfrak{H}_\alpha$ into $\mathfrak{H}$

$$\mathbf{U}_\alpha(\varepsilon)=\sum_\beta \hat{\mathbf{L}}_{\beta\alpha}(\varepsilon, \varepsilon). \tag{9.34}$$

Our next goal is to prove that the operators $\mathbf{U}_\alpha(\varepsilon)$ converge strongly on $\mathfrak{D}_\alpha$ for $\varepsilon\to 0$, and that the limit operators are isometric. For this we apply Hilbert's identity in the form (9.29). Multiplying (9.29) by

$$\overline{\psi_\alpha(k_\alpha)}\left(\frac{k^2}{2m}+\frac{p^2}{2n}-z_1\right)^{-1}\left(\frac{k'^2}{2m}+\frac{p'^2}{2n}-z_2\right)^{-1}\psi_\beta(k'_\beta)$$

and integrating over k_α and k'_β, we obtain with the help of (9.13) and (9.14)

$$\int \mathscr{L}_{\alpha\beta}(p_\alpha;\ k',\ p';\ z_1)\frac{\psi_\beta(k'_\beta)}{\frac{k'^2}{2m}+\frac{p'^2}{2n}-z_2}dk'_\beta\frac{z_2+\varkappa_\beta^2-\frac{p'^2_\beta}{2n_\beta}}{z_2-z_1}-$$

$$-\frac{z_1+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}}{z_2-z_1}\int\frac{\overline{\psi_\alpha(k_\alpha)}}{\frac{k_2}{2m}+\frac{p^2}{2n}-z_1}\mathscr{L}_{\alpha\beta}(k,\ p;\ p'_\beta;\ z_2)dk_\alpha=$$

$$=\int\sum_\gamma\mathscr{L}_{\alpha\gamma}(p_\alpha;\ k'',\ p'';\ z_1)\frac{1}{\frac{k''^2}{2m}+\frac{p''^2}{2n}-z_1}\,\frac{1}{\frac{k''^2}{2m}+\frac{p''^2}{2n}-z_2}\times$$

$$\times\sum_\gamma\mathscr{L}_{\gamma\beta}(k'',\ p'';\ p'_\beta;\ z_2)dk''dp''. \tag{9.35}$$

We set here

$$z_1=-\varkappa_\alpha^2+\frac{p_\alpha^2}{2n_\alpha}-i\varepsilon_1,\quad z_2=-\varkappa_\beta^2+\frac{p'^2_\beta}{2n_\beta}+i\varepsilon_2,$$

so that

$$z_2-z_1=-\varkappa_\beta^2+\frac{p'^2_\beta}{2n_\beta}+\varkappa_\alpha^2-\frac{p_\alpha^2}{2n_\alpha}+i(\varepsilon_1+\varepsilon_2),$$

multiply the result by $\overline{f'_\alpha(p_\alpha)}f_\beta(p'_\beta)$ and integrate with respect to p_α and p'_β. The right-hand member may be written

$$(\mathbf{U}_\beta(\varepsilon_2)f_\beta,\ \mathbf{U}_\alpha(\varepsilon_1)f_\alpha),$$

and it remains to consider the left-hand member.

The left-hand member contains for $\alpha=\beta$ a term which may be represented as

$$\frac{\delta(p_\alpha-p'_\alpha)}{i(\varepsilon_1+\varepsilon_2)}\left\{i\varepsilon_2\int\frac{\overline{\varphi_\alpha(k_\alpha)}\,\psi_\alpha(k_\alpha)}{\frac{k_\alpha^2}{2m_\alpha}+\varkappa_\alpha^2-i\varepsilon_2}dk_\alpha+i\varepsilon_1\int\frac{\overline{\psi_\alpha(k_\alpha)}\,\varphi_\alpha(k_\alpha)}{\frac{k_\alpha^2}{2m_\alpha}+\varkappa_\alpha^2+i\varepsilon_1}dk_\alpha\right\}\times$$

$$\times\overline{f'_\alpha(p_\alpha)}\,f_\alpha(p'_\alpha). \tag{9.36}$$

Here the integral

$$I(\varepsilon)=\int\frac{\overline{\varphi_\alpha(k)}\,\psi_\alpha(k)}{\frac{k^2}{2m_\alpha}+\varkappa_\alpha^2+i\varepsilon}dk$$

is analytic in ε near $\varepsilon=0$, while at $\varepsilon=0$ it reduces to the normalization integral for the function $\psi_\alpha(k)$, so that $I(0)=1$. Consequently, (9.36) may be written, after integration with respect to p_α and p'_α, in the form

$$(f_\alpha,\ f'_\alpha)_\alpha+O(\varepsilon_1)+O(\varepsilon_2).$$

The remaining terms of the left member may be expressed by means of integrals of the type

$$i\varepsilon_2\int\overline{f'_\alpha(p_\alpha)}\frac{\mathscr{K}_{\alpha\beta}\left(p_\alpha;\ k',\ p';\ -\varkappa_\alpha^2+\frac{p_\alpha^2}{2n_\alpha}-i\varepsilon_1\right)}{\varkappa_\beta^2-\frac{p'^2_\beta}{2n_\beta}-\varkappa_\alpha^2+\frac{p_\alpha^2}{2n_\alpha}-i(\varepsilon_1+\varepsilon_2)}\times$$

$$\times\frac{\psi_\beta(k'_\beta)}{\frac{k'^2_\beta}{2m_\beta}+\varkappa_\beta^2-i\varepsilon_2}f_\beta(p'_\beta)\,dp_\alpha dk'_\beta dp'_\beta. \tag{9.37}$$

Integrating first with respect to p'_β, we are left with the integral

$$k_{\alpha\beta}(p_\alpha, k'_\beta, \varepsilon_1, \varepsilon_2) = i\varepsilon_2 \int \frac{\mathcal{K}_{\alpha\beta}\left(p_\alpha; k', p'; -\varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha} - i\varepsilon_1\right)}{\varkappa_\beta^2 - \frac{p_\beta'^2}{2n_\beta} - \varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha} - i(\varepsilon_1 + \varepsilon_2)} f_\beta(p'_\beta)\, dp'_\beta. \tag{9.38}$$

We shall prove at the end of this section

Lemma 9.4. Let $f_\beta(p'_\beta) \in \mathfrak{N}(\theta, \mu)$ *and* $\frac{\varepsilon_1}{\varepsilon_2} > 0$. *Then the following estimate holds for* $k_{\alpha\beta}(p_\alpha, k'_\beta; \varepsilon_1, \varepsilon_2)$

$$|k_{\alpha\beta}(p_\alpha; k'_\beta; \varepsilon_1, \varepsilon_2)| \leqslant C(\varepsilon_2).$$

uniformly in all k'_β, $|\varepsilon_i| \leqslant 1$, $i = 1, 2$ *and for* p_α *varying within a region which does not include the singular surfaces* $-\varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha} = \lambda_n$, *whereby* $C(\varepsilon_2) \to 0$ *when* $\varepsilon_2 \to 0$.

Using this lemma and the preceding arguments, we may now prove

Lemma 9.5. The operators $\mathbf{U}_\alpha(\varepsilon)$, $\alpha = 23, 31, 12$ *converge strongly on* $\mathfrak{D}_\alpha$ *for* $\varepsilon \to 0$. *The limit operators*

$$\mathbf{U}_\alpha^{(\pm)} = \lim_{\varepsilon \to \mp 0} \mathbf{U}_\alpha(\varepsilon) \tag{9.39}$$

may be extended into isometric operators over the entire $\mathfrak{H}_\alpha$, *and the ranges of the operators* $\mathbf{U}_\alpha^{(\pm)}$ *and* $\mathbf{U}_\beta^{(\pm)}$ *are orthogonal for* $\alpha \neq \beta$, *i.e.*,

$$\mathbf{U}_\alpha^{(\pm)*}\mathbf{U}_\beta^{(\pm)} = \delta_{\alpha\beta}\mathbf{E}_\alpha. \tag{9.40}$$

Proof. If $f'_\alpha \in \mathfrak{D}_\alpha$, $f_\beta \in \mathfrak{D}_\beta$, then by Lemma 9.4 the integrals of the type (9.37) vanish when $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$; thus we have

$$(\mathbf{U}_\beta(\varepsilon_2)f_\beta, \mathbf{U}_\alpha(\varepsilon_1)f'_\alpha) = \delta_{\alpha\beta}(f_\alpha, f'_\alpha)_\alpha + o(1) \tag{9.41}$$

for $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, $\frac{\varepsilon_1}{\varepsilon_2} > 0$. Combining relations of the type (9.41), we conclude that if $f_\alpha \in \mathfrak{D}_\alpha$, then

$$\|[\mathbf{U}_\alpha(\varepsilon_1) - \mathbf{U}_\alpha(\varepsilon_2)]f_\alpha\| = o(1)$$

for $\varepsilon_1, \varepsilon_2 \to 0$, which implies the strong convergence of $\mathbf{U}_\alpha(\varepsilon)$ on $\mathfrak{D}_\alpha$. Setting $\varepsilon_1 = \varepsilon_2 = \varepsilon$ in (9.41), we pass to the limit for $\varepsilon \to 0$, and obtain for any $f_\alpha \in \mathfrak{D}_\alpha$, $f_\beta \in \mathfrak{D}_\beta$

$$(\mathbf{U}_\alpha^{(\pm)}f_\alpha, \mathbf{U}_\beta^{(\pm)}f_\beta) = \delta_{\alpha\beta}(f_\alpha, f'_\alpha)_\alpha.$$

It follows in view of the denseness of $\mathfrak{D}_\alpha$ in $\mathfrak{H}_\alpha$ that the operators $\mathbf{U}_\alpha^{(\mp)}$ are extended by closure over the entire $\mathfrak{H}_\alpha$, $\alpha = 23, 31, 12$, and (9.40) follows. This completes the proof.

We now go on to prove that the ranges of the operators $\mathbf{U}_0^{(\pm)}$ and $\mathbf{U}_\alpha^{(\pm)}$, $\alpha = 23, 31, 12$, are also orthogonal. To this end we apply again Hilbert's identity. Multiplying (9.29) by $\left(\frac{k'^2}{2m} + \frac{p'^2}{2n} - z_2\right)^{-1} \psi_\beta(k'_\beta)$, integrating with respect to k'_β, and using (9.13),

$$\int \mathcal{M}_{\alpha\beta}(k, p; k', p', z_1) \frac{\psi_\beta(k'_\beta)}{\frac{k'^2}{2m} + \frac{p'^2}{2n} - z_2} dk'_\beta \frac{z_2 + \varkappa_\beta^2 - \frac{p_\beta'^2}{2n_\beta}}{z_2 - z_1} = \frac{\mathcal{L}_{\alpha\beta}(k, p; p'_\beta; z_2)}{z_2 - z_1} +$$

$$+ \int \sum_\gamma \mathcal{M}_{\alpha\gamma}(k, p; k'', p''; z_1) \frac{1}{\frac{k''^2}{2m} + \frac{p''^2}{2n} - z_1} \frac{1}{\frac{k''^2}{2m} + \frac{p''^2}{2n} - z_2} \sum_\gamma \mathcal{L}_{\gamma\beta}(k'', p''; p'_\beta; z_2)\, dk''dp''. \tag{9.42}$$

We set here

$$z_1 = \frac{k^2}{2m} + \frac{p^2}{2n} - i\varepsilon, \quad z_2 = -\varkappa_\beta^2 + \frac{p_\beta'^2}{2n_\beta} + i\varepsilon,$$

so that

$$z_2 - z_1 = -\left(\frac{k^2}{2m} + \frac{p^2}{2n} + \varkappa_\beta^2 - \frac{p_\beta'^2}{2n_\beta} - 2i\varepsilon\right),$$

multiply the result by $\overline{f_0'(k, p)}\, f_\beta(p_\beta')$, and integrate with respect to k, p and p_β'. After summation over α and β, the right-hand member may be written

$$(\mathbf{U}_\beta(\varepsilon) f_\beta,\ \mathbf{U}_0(\varepsilon) f_0) + o(1).$$

In order to obtain an expression for the left-hand member, it is necessary to evaluate the integral

$$m_{\alpha\beta}(k, p;\, k_\beta';\, \varepsilon) = i\varepsilon \int \frac{\mathcal{M}_{\alpha\beta}\left(k, p;\, k', p';\, \frac{k^2}{2m} + \frac{p^2}{2n} - i\varepsilon\right)}{\frac{k^2}{2m} + \frac{p^2}{2n} + \varkappa_\beta^2 - \frac{p_\beta'^2}{2n_\beta} - 2i\varepsilon} f_\beta(p_\beta')\, dp_\beta', \tag{9.43}$$

multiply the result by

$$\overline{f_0(k, p)}\, \varphi_\beta(k_\beta') \left(\frac{k_\beta'^2}{2m_\beta} + \varkappa_\beta^2 + i\varepsilon\right)^{-1},$$

integrate with respect to k, p, k_β', and sum over α and β. At the end of this section we shall prove

Lemma 9.6. Let $f_\beta(p_\beta) \in \mathfrak{N}(\theta, \mu)$. *Then* $m_{\alpha\beta}(k, p;\, k_\beta';\, \varepsilon)$ *satisfies the estimate*

$$|m_{\alpha\beta}(k, p;\, k_\beta';\, \varepsilon)| \leqslant C(\varepsilon),$$

uniformly in ε, $|\varepsilon| \leqslant 1$, k_β' *and* k, *and* p *lying in a region which does not contain the singular surfaces* $\frac{k^2}{2m} + \frac{p^2}{2n} = \lambda_\varkappa$, *whereby* $C(\varepsilon) \to 0$ *when* $\varepsilon \to 0$.

This lemma leads to

Lemma 9.7. The ranges of the operators $\mathbf{U}_0^{(\pm)}$ *and* $\mathbf{U}_\beta^{(\pm)}$, $\beta = 23, 31, 12$, *are orthogonal, i.e.,*

$$\mathbf{U}_0^{(\pm)*} \mathbf{U}_\beta^{(\pm)} = \mathbf{U}_\beta^{(\pm)*} \mathbf{U}_0^{(\pm)} = 0. \tag{9.44}$$

Proof. From Lemma 9.6 and the preceding arguments we know that if $f_0 \in \mathfrak{D}_0$ and $f_\beta \in \mathfrak{D}_\beta$, then

$$(\mathbf{U}_\beta(\varepsilon) f_\beta,\ \mathbf{U}_0(\varepsilon) f_0) = o(1)$$

for $\varepsilon \to 0$. Passing here to the limit for $\varepsilon \to \pm 0$, we deduce that $(\mathbf{U}_\beta^{(\pm)} f_\beta, \mathbf{U}_0^{(\pm)} f_0) = 0$, for such f_0 and f_β. This relation is extended by closure througnout $\mathfrak{H}_\beta$ and $\mathfrak{H}_0$, whence (9.44). This completes the proof.

We have thus given an exact meaning to the formal expressions (9.3) and (9.5) for the functions $\Psi_0^{(\pm)}(k, p;\, k^{(0)}, p^{(0)})$ and $\Psi_\alpha^{(\pm)}(k, p;\, p_\alpha^{(0)})$ and proved the orthogonality of these functions. Let us now prove that these functions are eigenfunctions of the operator $\mathbf{H}$. We denote by $\tilde{\mathbf{R}}_0(z)$ and $\tilde{\mathbf{R}}_\alpha(z)$ the resolvents of $\tilde{\mathbf{H}}_0$ and $\tilde{\mathbf{H}}_\alpha$, operating in the spaces $\mathfrak{H}_0$ and $\mathfrak{H}_\alpha$, $\alpha = 23, 31, 12$, respectively.

Lemma 9.8. The following relations hold

$$\mathbf{R}(z)\, \mathbf{U}_0^{(\pm)} = \mathbf{U}_0^{(\pm)} \tilde{\mathbf{R}}_0(z); \tag{9.45}$$

$$\mathbf{R}(z)\, \mathbf{U}_\beta^{(\pm)} = \mathbf{U}_\beta^{(\pm)} \tilde{\mathbf{R}}_\beta(z), \quad \beta = 23, 31, 12. \tag{9.46}$$

Proof. We set in (9.29)

$$z_1 = z \quad \text{and} \quad z_2 = \frac{k'^2}{2m} + \frac{p'^2}{2n} + i\varepsilon,$$

so that

$$z_2 - z_1 = \frac{k'^2}{2m} + \frac{p'^2}{2n} - z + i\varepsilon.$$

We write the denominator $(z_2 - z_1)^{-1}$ in the second term on the left-hand side of the obtained relation in the form

$$-\frac{1}{\frac{k^2}{2m} + \frac{p^2}{2n} - \frac{k'^2}{2m} - \frac{p'^2}{2n} - i\varepsilon} + \left(\frac{k^2}{2m} + \frac{p^2}{2n} - z\right) \times$$
$$\times \left(\frac{1}{\frac{k^2}{2m} + \frac{p^2}{2n} - \frac{k'^2}{2m} - \frac{p'^2}{2n} - i\varepsilon} \cdot \frac{1}{\frac{k'^2}{2m} + \frac{p'^2}{2n} - z + i\varepsilon}\right)$$

and obtain a relation which is equivalent to the operator equation

$$\mathbf{M}_{\alpha\beta}(z)\mathbf{R}_0(z - i\varepsilon)\hat{\mathbf{E}}_0 + \hat{\mathbf{M}}_{\alpha\beta}(\varepsilon, \varepsilon) - (\mathbf{H}_0 - z\mathbf{E})\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon, \varepsilon)\tilde{\mathbf{R}}_0(z - i\varepsilon) =$$
$$= \sum_{\gamma} \mathbf{M}_{\alpha\gamma}(z)\mathbf{R}_0(z)\sum_{\gamma} \hat{\mathbf{M}}_{\gamma\beta}(\varepsilon, \varepsilon). \tag{9.47}$$

Multiplying (9.47) by $\mathbf{R}_0(z)$, adding and subtracting the identity

$$\mathbf{R}_0(z)\hat{\mathbf{E}}_0 = \hat{\mathbf{E}}_0\tilde{\mathbf{R}}_0(z),$$

summing over α and β, and rearranging terms, we obtain

$$\left[\mathbf{R}_0(z) - \mathbf{R}_0(z)\sum_{\alpha,\beta} \mathbf{M}_{\alpha\beta}(z)\mathbf{R}_0(z - i\varepsilon)\right]\left[\hat{\mathbf{E}}_0 - \sum_{\alpha,\beta} \hat{\mathbf{M}}_{\alpha\beta}(\varepsilon, \varepsilon)\right] =$$
$$= \hat{\mathbf{E}}_0\tilde{\mathbf{R}}_0(z) - \sum_{\alpha,\beta} \hat{\mathbf{M}}_{\alpha\beta}(\varepsilon, \varepsilon)\tilde{\mathbf{R}}_0(z - i\varepsilon).$$

Recalling (9.1), (9.26), and passing to the limit for $\varepsilon \to 0$, we obtain (9.45).

We now set in (9.42)

$$z_1 = z, \quad z_2 = -\varkappa_\beta^2 + \frac{p_\beta'^2}{2n_\beta} + i\varepsilon,$$

so that

$$z_2 - z_1 = -\varkappa_\beta^2 + \frac{p_\beta'^2}{2n_\beta} - z + i\varepsilon.$$

We write the denominator $(z_2 - z_1)^{-1}$ in the form

$$-\frac{1}{\frac{k^2}{2m} + \frac{p^2}{2n} + \varkappa_\beta^2 - \frac{p_\beta'^2}{2n_\beta} - i\varepsilon} + \left(\frac{k^2}{2m} + \frac{p^2}{2n} - z\right) \times$$
$$\times \left(\frac{1}{\frac{k^2}{2m} + \frac{p^2}{2n} + \varkappa_\beta^2 - \frac{p_\beta'^2}{2n_\beta} - i\varepsilon}\,\frac{1}{-\varkappa_\beta^2 + \frac{p_\beta'^2}{2n_\beta} - z + i\varepsilon}\right).$$

The right-hand member of the resulting expression may be written

$$-\hat{\mathbf{L}}_{\alpha\beta}(\varepsilon, \varepsilon) + (\mathbf{H}_0 - z\mathbf{E})\hat{\mathbf{L}}_{\alpha\beta}(\varepsilon, \varepsilon)\tilde{\mathbf{R}}_\beta(z - i\varepsilon) + \sum_{\gamma} \mathbf{M}_{\alpha\gamma}(z)\mathbf{R}_0(z)\sum_{\gamma} \hat{\mathbf{L}}_{\gamma\beta}(\varepsilon, \varepsilon).$$

In the left-hand member we have a nonsingular expression, containing the factor ε. We multiply the last relation by $\mathbf{R}_0(z)$ and sum over α. Passing to the limit for $\varepsilon \to \pm 0$ gives (9.46), which completes the proof.

We obtain as a corollary to this lemma, that if the function $\varphi(x)$ is bounded for any x, $-\infty < x < \infty$ then

$$\varphi(\mathbf{H})\mathbf{U}_0^{(\pm)} = \mathbf{U}_0^{(\pm)}\varphi(\tilde{\mathbf{H}}_0); \tag{9.48}$$

$$\varphi(\mathbf{H})\mathbf{U}_\alpha^{(\pm)} = \mathbf{U}_\alpha^{(\pm)}\varphi(\tilde{\mathbf{H}}_\alpha). \tag{9.49}$$

We may therefore say that the kernels of the operators $\mathbf{U}_0^{(\pm)}$ and $\mathbf{U}_\alpha^{(\pm)}$, i.e., the functions

$$\Psi_0^{(\pm)}(k,\ p;\ k^{(0)},\ p^{(0)}) \text{ and } \Psi_\alpha^{(\pm)}(k,\ p;\ p_\alpha^{(0)}),$$

are in a certain sense solutions of the equation

$$\mathbf{H}\Psi_0^{(\pm)}(k,\ p;\ k^{(0)},\ p^{(0)}) = \left(\frac{k^{(0)^2}}{2m} + \frac{p^{(0)^2}}{2n}\right)\Psi_0^{(\pm)}(k,\ p;\ k^{(0)},\ p^{(0)});$$

$$\mathbf{H}\Psi_\alpha^{(\pm)}(k,\ p;\ p_\alpha^{(0)}) = \left(-\varkappa_\alpha^2 + \frac{p_\alpha^{(0)^2}}{2n_\alpha}\right)\Psi_\alpha^{(\pm)}(k,\ p;\ p_\alpha^{(0)}),$$

so that they may be regarded as eigenfunctions of the operator $\mathbf{H}$.

Our next problem is to prove the completeness of the derived system of eigenfunctions. For this we will first derive, as in § 8, a convenient representation for the spectral function $\mathbf{E}(\lambda)$ of $\mathbf{H}$.

We introduce the notations

$$m_{\alpha\beta}(k,\ p;\ z;\ f) = m_{\alpha\beta}(k,\ p;\ z_1,\ z_2;\ f)|_{z_1=z_2=z}; \tag{9.50}$$

$$l_{\alpha\beta}(p_\alpha;\ z;\ f) = l_{\alpha\beta}(p_\alpha;\ z_1,\ z_2;\ f)|_{z_1=z_2=z}; \tag{9.51}$$

(for the definition of the right-hand sides cf. (9.17), (9.19)), and

$$g_0^{(\pm)}(k,\ p;\ f) = f(k,\ p) + \sum_{\alpha,\beta} m_{\alpha\beta}\left(k,\ p;\ \frac{k^2}{2m} + \frac{p^2}{2n} \pm i0; f\right); \tag{9.52}$$

$$g_\alpha^{(\pm)}(p_\alpha;\ f) = \sum_\beta l_{\alpha\beta}\left(p_\alpha;\ -\varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha} \pm i0;\ f\right). \tag{9.53}$$

By Lemma 9.1 these functions are defined if $\frac{k^2}{2m} + \frac{p^2}{2n} \neq \lambda_*$, and $-\varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha} \neq \lambda_*$, respectively.

Lemma 9.9. If λ *is not a singular point of the operator* $\mathbf{A}(z)$, *and if* $f,\ f' \in \mathfrak{M}(\theta,\ \mu)$, *then* $(\mathbf{E}(\lambda)f,\ f')$ *is differentiable with respect to* λ, *and*

$$\frac{d}{d\lambda}(\mathbf{E}(\lambda)f,\ f') = \int g_0^{(\pm)}(k,\ p;\ f)\overline{g_0^{(\pm)}(k,\ p;\ f')}\,\delta\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda\right)dk\,dp +$$

$$+ \sum_\alpha \int g_\alpha^{(\pm)}(p_\alpha;\ f)\overline{g_\alpha^{\pm}(p_\alpha;\ f')}\,\delta\left(-\varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha} - \lambda\right)dp_\alpha. \tag{9.54}$$

Proof. We make use of the relation between the spectral function and the resolvent

$$\int_\mu^\nu d(\mathbf{E}(\lambda)f,\ f') = \lim_{\varepsilon \to +0}\frac{1}{2\pi i}\int_\mu^\nu [(\mathbf{R}(\lambda + i\varepsilon) - \mathbf{R}(\lambda - i\varepsilon)]f,\ f')\,d\lambda. \tag{9.55}$$

In virtue of (9.1) the difference between the resolvents on the right-hand side of (9.55) may be represented as

$$\mathbf{R}(\lambda + i\varepsilon) - \mathbf{R}(\lambda - i\varepsilon) = \left[\mathbf{E} - \mathbf{R}_0(\lambda + i\varepsilon)\sum_{\alpha,\beta}\mathbf{M}_{\alpha\beta}(\lambda + i\varepsilon)\right] \times$$

$$\times (\mathbf{R}_0(\lambda + i\varepsilon) - \mathbf{R}_0(\lambda - i\varepsilon))\left[\mathbf{E} - \sum_{\alpha',\beta'}\mathbf{M}_{\alpha'\beta'}(\lambda - i\varepsilon)\mathbf{R}_0(\lambda - i\varepsilon)\right]. \tag{9.56}$$

Let us construct the quadratic form associated with this expression for elements f and f' in $\mathfrak{M}(\theta, \mu)$. The right-hand side is a sum of integrals of the type

$$\int \left\{ n_{\alpha\beta}(k, p; \lambda + i\varepsilon; f) + \frac{\varphi_\alpha(k_\alpha) l_{\alpha\beta}(p_\alpha; \lambda + i\varepsilon; f)}{\lambda + i\varepsilon + \varkappa_\alpha^2 - \frac{p_\alpha^2}{2n_\alpha}} \right\} \times$$
$$\times \left[\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda - i\varepsilon \right)^{-1} - \left(\frac{k^2}{2m} + \frac{p^2}{2n} - \lambda + i\varepsilon \right)^{-1} \right] \times$$
$$\times \left\{ \overline{n_{\alpha'\beta'}(k, p; \lambda - i\varepsilon; f')} + \frac{\overline{\varphi_{\alpha'}(k_{\alpha'})}\, \overline{l_{\alpha'\beta'}(p_{\alpha'}; \lambda - i\varepsilon; f')}}{\lambda - i\varepsilon + \varkappa_{\alpha'}^2 - \frac{p_{\alpha'}^2}{2n_{\alpha'}}} \right\} dkdp.$$

We have written out here the integral which results from the multiplication of the second terms in the square brackets on the right-hand side of (9.56), and used the notations of (9.17), (9.18), (9.50) and (9.51), and the symmetry relation (9.23) for the kernels $\mathcal{M}_{\alpha\beta}(k, p; k', p'; z)$. Such integrals have in fact been considered before in § 7. By the same arguments which led to the proof of Lemma 7.9, we may prove that

$$\frac{1}{2\pi i}([\mathbf{R}(\lambda + i\varepsilon) - \mathbf{R}(\lambda - i\varepsilon)]f, f')$$

tends to a limit for $\varepsilon \to +0$, and that this limit is identical with the right-hand side of (9.54). This completes the proof.

Let us split the spectral function $\mathbf{E}(\lambda)$ into the sum of two terms: the step function $\mathbf{E}_d(\lambda)$ and the continuous function $\mathbf{E}_c(\lambda)$. The complete course of the function $\mathbf{E}_d(\lambda)$ provides the projection operator $\mathbf{P}_d$ on the subspace spanned by the eigenfunctions of the discrete spectrum of $\mathbf{H}$. It is also clear that

$$\int_{-\infty}^{\infty} d\mathbf{E}_c(\lambda) = \mathbf{E} - \mathbf{P}_d \tag{9.57}$$

and that $\mathbf{E}(\lambda)$ on the left-hand side of (9.54) may be replaced by $\mathbf{E}_c(\lambda)$.

Lemma 9.10. The following relation is valid

$$\mathbf{U}_0^{(\pm)}\mathbf{U}_0^{(\pm)*} + \mathbf{U}_{23}^{(\pm)}\mathbf{U}_{23}^{(\pm)*} + \mathbf{U}_{31}^{(\pm)}\mathbf{U}_{31}^{(\pm)*} + \mathbf{U}_{12}^{(\pm)}\mathbf{U}_{12}^{(\pm)*} = \mathbf{E} - \mathbf{P}_d. \tag{9.58}$$

Proof. We first show that the functions $g_0^{(\pm)}(k, p; f)$ and $g_\alpha^{(\pm)}(p_\alpha; f)$ are square-integrable over the entire six-dimensional and three-dimensional spaces, respectively. We denote by l_ε the real axis slit along the intervals $[\lambda_* - \varepsilon, \lambda_* + \varepsilon]$ and by $\Omega_\varepsilon^{(0)}$ and $\Omega_\varepsilon^{(\alpha)}$ the one six-dimensional space, and the three three-dimensional spaces slit along the spherical shells $\left| \frac{k^2}{2m} + \frac{p^2}{2n} - \lambda_* \right| \leqslant \varepsilon$ and $\left| -\varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha} - \lambda_* \right| \leqslant \varepsilon$, respectively. Setting $f = f'$ in (9.54) and integrating both sides with respect to λ over l_ε, we obtain

$$\int_{l_\varepsilon} \frac{d}{d\lambda}(\mathbf{E}_c(\lambda)f, f)\, d\lambda = \int_{\Omega_\varepsilon^{(0)}} |g_0^{(\pm)}(k, p; f)|^2 dkdp +$$
$$+ \sum_\alpha \int_{\Omega_\varepsilon^{(\alpha)}} |g_\alpha^{(\pm)}(p_\alpha; f)|^2 dp_\alpha. \tag{9.59}$$

The integrals on both sides are monotonic in ε and do not decrease with decreasing ε. For $\varepsilon \to 0$, the left member of (9.59) converges to $([\mathbf{E} - \mathbf{P}_d]f, f)$ while the integrals on the right member extend over the entire six- or three-dimensional space, respectively. We deduce that

$$\int |g_0^{(\pm)}(k, p; f)|^2 dkdp + \sum_\alpha \int |g_\alpha^{(\pm)}(p_\alpha; f)|^2 dp_\alpha =$$
$$= ([\mathbf{E} - \mathbf{P}_d]f, f) \leqslant (f, f), \tag{9.60}$$

which implies, in particular, the required square-integrability.

We now denote by $g_0^{(\pm)}(f)$ and $g_\alpha^{(\pm)}(f)$ the elements of $\mathfrak{H}_0$ and $\mathfrak{H}_\alpha$, generated by the functions $g_0^{(\pm)}(k, p; f)$ and $g_\alpha^{(\pm)}(p_\alpha; f)$. Let us show that

$$g_0^{(\pm)}(f) = \mathbf{U}_0^{(\pm)*}f; \quad g_\alpha^{(\pm)}(f) = \mathbf{U}_\alpha^{(\pm)*}f. \tag{9.61}$$

We note that in virtue of (9.22) the following relation holds for any $f_0 \in \mathfrak{D}_0$ and $f \in \mathfrak{M}(\theta, \mu)$

$$(\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon, \varepsilon)f_0, f) = \int f_0(k, p)\overline{m_{\beta\alpha}\left(k, p; \frac{k^2}{2m} + \frac{p^2}{2n} - i\varepsilon; f\right)}dkdp.$$

We sum this relation over α and β and add to both sides the quantity

$$(\hat{\mathbf{E}}_0 f_0, f) = \int f_0(k, p)\overline{f(k, p)}\,dkdp.$$

The result is

$$(\mathbf{U}_0(\varepsilon)f_0, f) = \int f_0(k, p)\overline{g_0\left(k, p; \frac{k^2}{2m} + \frac{p^2}{2n} - i\varepsilon; f\right)}dkdp.$$

Passing to the limit for $\varepsilon \to \mp 0$, we obtain

$$(\mathbf{U}_0^{(\pm)}f_0, f) = (f_0, g_0^{(\pm)}(f))_0,$$

which, in view of the denseness of $\mathfrak{D}_0$ in $\mathfrak{H}_0$, entails the first of the relations (9.61).

The second of these relations is proved analogously. Let $f_\alpha \in \mathfrak{D}_\alpha$ and $f \in \mathfrak{M}(\theta, \mu)$. In view of (9.24)

$$(\mathbf{L}_{\beta\alpha}(\varepsilon, \varepsilon)f_\alpha, f) = \int f_\alpha(p_\alpha)\overline{l_{\alpha\beta}\left(p_\alpha; -\varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha} - i\varepsilon; f\right)}dp_\alpha.$$

Summing over β and passing to the limit for $\varepsilon \to \mp 0$,

$$(\mathbf{U}_\alpha^{(\pm)}f_\alpha, f) = (f_\alpha, g_\alpha^{(\pm)}(f))_\alpha,$$

which, in view of the denseness of $\mathfrak{D}_\alpha$ in $\mathfrak{H}_\alpha$, entails the second relation of (9.61).

Replacing $\mathbf{E}(\lambda)$ in (9.54) by $\mathbf{E}_c(\lambda)$, integrating the result with respect to λ from $-\infty$ to ∞, and applying (9.61) and (9.57), we obtain

$$((\mathbf{E} - \mathbf{P}_d)f, f') = (\mathbf{U}_0^{(\pm)*}f, \mathbf{U}_0^{(\pm)*}f') + (\mathbf{U}_{23}^{(\pm)*}f, \mathbf{U}_{23}^{(\pm)*}f') +$$
$$+ (\mathbf{U}_{31}^{(\pm)*}f, \mathbf{U}_{31}^{(\pm)*}f') + (\mathbf{U}_{12}^{(\pm)*}f, \mathbf{U}_{12}^{(\pm)*}f'),$$

whence (9.58), which completes the proof.

The preceding results entail

Theorem 9.2. *There exist operators* $\mathbf{U}_0^{(\pm)}$ *and* $\mathbf{U}_\alpha^{(\pm)}$, $\alpha = 23, 31, 12$, *respectively mapping* $\mathfrak{H}_0$ *and* $\mathfrak{H}_\alpha$ *into* $\mathfrak{H}$, *which possess the following properties:*

1. Any function $f \in \mathfrak{H}$ *is uniquely represented in the form*

$$f = f_d + \mathbf{U}_0^{(\pm)}f_0^{(\pm)} + \mathbf{U}_{23}^{(\pm)}f_{23}^{(\pm)} + \mathbf{U}_{31}^{(\pm)}f_{31}^{(\pm)} + \mathbf{U}_{12}^{(\pm)}f_{12}^{(\pm)}, \tag{9.62}$$

where

$$f_d \in \mathfrak{H}_d, \; f_0^{(\pm)} \in \mathfrak{H}_0, \; f_\alpha^{(\pm)} \in \mathfrak{H}_\alpha, \; \alpha = 23, 31, 12.$$

2. The following relation holds for any function $\varphi(x)$, *bounded over the entire real axis*

$$\varphi(\mathbf{H})f=\varphi(\mathbf{P}_d\mathbf{H})f_d+\mathbf{U}_0^{(\pm)}\varphi(\hat{\mathbf{H}}_0)f_0^{(\pm)}+\sum\mathbf{U}_\alpha^{(\pm)}\varphi(\hat{\mathbf{H}}_\alpha)f_\alpha^{(\pm)}. \tag{9.63}$$

3. The functions $f_0^{(\pm)}$, $f_\alpha^{(\pm)}$ *are defined by*

$$f_0^{(\pm)}=\mathbf{U}_0^{(\pm)*}f;\quad f_\alpha^{(\pm)}=\mathbf{U}_\alpha^{(\pm)*}f. \tag{9.64}$$

4. The following relation holds

$$\|f\|^2=\|f_d\|^2+\|f_0^{(\pm)}\|_0^2+\sum_\alpha\|f_\alpha^{(\pm)}\|_\alpha^2. \tag{9.65}$$

Formulas (9.62) and (9.64) constitute a concise and rigorous statement of the expansion theorem for an arbitrary function in the eigenfunctions of the operator $\mathbf{H}$. The equality (9.65) corresponds to the Parseval relation.

Theorem 9.1 also follows from Theorem 9.2. We set for $\mathbf{U}$ in Theorem 9.1 the operators $\mathbf{U}^{(\pm)}$, defined as follows: if $\hat{f}\in\hat{\mathfrak{H}}$ and f_0, f_{23}, f_{31}, f_{12} are the components of this element, then

$$\mathbf{U}^{(\pm)}\hat{f}=\mathbf{U}_0^{(\pm)}f_0+\mathbf{U}_{23}^{(\pm)}f_{23}+\mathbf{U}_{31}^{(\pm)}f_{31}+\mathbf{U}_{12}^{(\pm)}f_{12}. \tag{9.66}$$

It is then easily verified that in virtue of Theorem 9.2 the operators $\mathbf{U}^{(\pm)}$ map isometrically $\hat{\mathfrak{H}}$ onto $\mathfrak{H}_c$ and possess the properties listed in Theorem 9.1.

Consider in $\hat{\mathfrak{H}}$ the operator

$$\mathbf{S}=\mathbf{U}^{(+)*}\mathbf{U}^{(-)}. \tag{9.67}$$

If $\hat{f}$, $\hat{f}'\in\hat{\mathfrak{H}}$, and f_0, f_α, f_0', f_α', $\alpha=23,\ 31,\ 12$, are the components of $\hat{f}$ and $\hat{f}'$, respectively, then

$$\hat{f}'=\mathbf{S}\hat{f}$$

means that

$$f_\alpha'=\sum_\beta\mathbf{S}_{\alpha\beta}f_\beta.$$

Here summation is carried out over the values $\beta=0,\ 23,\ 31,\ 12$, and the $\mathbf{S}_{\alpha\beta}$ are operators that map $\mathfrak{H}_\beta$ into $\mathfrak{H}_\alpha$, defined by

$$\mathbf{S}_{\alpha\beta}=\mathbf{U}_\alpha^{(+)*}\mathbf{U}_\beta^{(-)},\quad \alpha,\ \beta=0,\ 23,\ 31,\ 12. \tag{9.68}$$

Lemma 9.11. *The operator* $\mathbf{S}$ *is unitary and commutes with any bounded function of the operator* $\hat{\mathbf{H}}$. *The following relations are valid*

$$\mathbf{U}^{(-)}=\mathbf{U}^{(+)}\mathbf{S}, \tag{9.69}$$

that is,

$$\mathbf{U}_\alpha^{(-)}=\sum_\beta\mathbf{U}_\beta^{(+)}\mathbf{S}_{\beta\alpha},\quad \alpha,\ \beta=0,\ 23,\ 31,\ 12. \tag{9.70}$$

The proof corresponds almost exactly to that of Lemma 8.8. The fact that now the operators $\mathbf{U}^{(\pm)}$ relate to different spaces is immaterial.

The operators $\mathbf{S}_{\alpha\beta}$ may be explicitly expressed by means of the kernels $\mathcal{M}_{\alpha\beta}(k,\ p;\ k',\ p';\ z)$. We derive as an example one such expression, without justifying rigorously all the steps.

We substitute $\varepsilon_2=-\varepsilon$ and $\varepsilon_1=2\varepsilon$ in (9.31); then

$$(\mathbf{U}_0(-\varepsilon)f_0,\ \mathbf{U}_0(2\varepsilon)f_0')=(f_0,\ f_0')_0+\Big(\sum_{\alpha,\beta}[\mathbf{M}_{\alpha\beta}(-\varepsilon,\ \varepsilon)-\mathbf{M}_{\alpha\beta}(-\varepsilon,\ -\varepsilon)]f_0,\ \hat{\mathbf{E}}_0f_0'\Big).$$

Passing here to the limit for $\varepsilon \to -0$, we obtain

$$\mathbf{S}_{00} f_0 = \{ \mathbf{E}_0 + \hat{\mathbf{E}}_0^* \sum_{\alpha, \beta} [\hat{\mathbf{M}}_{\alpha\beta}(+0, -0) - \hat{\mathbf{M}}_{\alpha\beta}(+0, +0)] \} f_0. \tag{9.71}$$

The right-hand side converges, at least weakly, on a dense set. The kernel of $\mathbf{S}_{00}$, by (9.71) and the definition (9.15) of the operators $\hat{\mathbf{M}}_{\alpha\beta}(\varepsilon_1, \varepsilon_2)$, may be formally written as

$$\mathscr{S}_{00}(k, p; k', p') = \delta(k - k')\delta(p - p') -$$
$$- 2\pi i \delta\left(\frac{k^2}{2m} + \frac{p^2}{2n} - \frac{k'^2}{2m} - \frac{p'^2}{2n}\right) \sum_{\alpha, \beta} \mathscr{M}_{\alpha\beta}\left(k, p; k', p'; \frac{k'^2}{2m} + \frac{p'^2}{2n} + i0\right).$$

We write down without derivation the expressions for the kernels of the other $\mathbf{S}_{\alpha\beta}$:

$$\mathscr{S}_{0\alpha}(k, p; p'_\alpha) = -2\pi i \delta\left(\frac{k^2}{2m} + \frac{p^2}{2n} + \varkappa_\alpha^2 - \frac{p'^2_\alpha}{2n_\alpha}\right) \times$$
$$\times \sum_{\beta} \mathscr{K}_{\beta\alpha}\left(k, p; p'_\alpha; -\varkappa_\alpha^2 + \frac{p'^2_\alpha}{2n_\alpha} + i0\right),$$
$$\mathscr{S}_{\alpha\beta}(p_\alpha; p'_\beta) = \delta_{\alpha\beta}\delta(p_\alpha - p'_\alpha) -$$
$$- 2\pi i \delta\left(-\varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha} + \varkappa_\beta^2 - \frac{p'^2_\beta}{2n_\beta}\right) \mathscr{H}_{\alpha\beta}\left(p_\alpha; p'_\beta; -\varkappa_\beta^2 + \frac{p'^2_\beta}{2n_\beta} + i0\right).$$

We conclude this section with proof of Lemmas 9.4 and 9.6. We encounter here integrals of a somewhat more general type than those actually required in the lemmas, namely

$$w_{\alpha\beta}(k, p; k'_\beta; z_1, z_2; f) = (z_2 - z_1) \int \frac{\mathscr{W}_{\alpha\beta}(k, p; k', p'; z_1)}{z_2 + \varkappa_\beta^2 - \dfrac{p'^2_\beta}{2n_\beta}} f_\beta(p'_\beta)\, dp'_\beta. \tag{9.72}$$

According to a formula of the type (5.3) for the kernel $\mathscr{W}_{\alpha\beta}$, these integrals may be represented in the form

$$w_{\alpha\beta}(k, p; z_1, z_2; k'_\beta) = \rho_{\alpha\beta}(k, p; z_1, z_2; k'_\beta) +$$
$$+ \frac{\varphi_\alpha(k_\alpha)\, \sigma_{\alpha\beta}(p_\alpha; z_1, z_2; k'_\beta)}{z_1 + \varkappa_\alpha^2 - \dfrac{p_\alpha^2}{2n_\alpha}}, \tag{9.73}$$

where, in particular,

$$\sigma_{\alpha\beta}(p_\alpha; z_1, z_2; k'_\beta) = (z_2 - z_1) \int \frac{\mathscr{K}_{\alpha\beta}(p_\alpha; k', p'; z_1)}{z_2 + \varkappa_\beta^2 - \dfrac{p'^2_\beta}{2n_\beta}} f_\beta(p'_\beta)\, dp'_\beta. \tag{9.74}$$

Comparing (9.43), (9.38), and (9.72), (9.74), and using (9.4), we find that the integrals (9.43) and (9.38) are expressed by those introduced in (9.72), as follows

$$m_{\alpha\beta}(k, p; k'_\beta; \varepsilon) = i\varepsilon t_\alpha\left(k_\alpha, k'_\alpha, \frac{k'^2_\alpha}{2m_\alpha} + i\varepsilon\right) f_\beta(p'_\beta)\delta_{\alpha\beta} +$$
$$+ w_{\alpha\beta}\left(k, p; \frac{k^2}{2m} + \frac{p^2}{2n} + i\varepsilon; \frac{k^2}{2m} + \frac{p^2}{2n} + 2i\varepsilon; k'_\beta\right);$$
$$k_{\alpha\beta}(p_\alpha, k'_\beta, \varepsilon_1, \varepsilon_2) =$$
$$= \sigma_{\alpha\beta}\left(p_\alpha; -\varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha} + i\varepsilon_1; -\varkappa_\alpha^2 + \frac{p_\alpha^2}{2n_\alpha} + i(\varepsilon_1 + \varepsilon_2); k'_\beta\right).$$

Let us now prove

Lemma 9.12. Let $f_\beta \in \mathfrak{N}(\theta, \mu)$. *Then the functions* $\rho_{\alpha\beta}$ *and* $\sigma_{\alpha\beta}$ *satisfy the estimates*

$$|\rho_{\alpha\beta}(k, p;\ z_1, z_2;\ k'_\beta)| \leqslant C(|z_1 - z_2|);$$
$$|\sigma_{\alpha\beta}(p_\alpha;\ z_1, z_2;\ k'_\beta)| \leqslant C(|z_1 - z_2|),$$

uniformly in any k'_β, *for all* z_1 *varying within any finite region of the plane* $\Pi_{-\varkappa^2}$ *that does not contain singular points, and for all* z_2 *varying in that region and satisfying the condition* $|z_1 - z_2| \leqslant \delta$, *where* $\delta \leqslant \frac{1}{2}\min\limits_\alpha \varkappa_\alpha^2$. *Whereby* $C(|z_1 - z_2|) \to 0$ *when* $z_1 \to z_2$.

Proof. We apply the same method as in the proof of Lemma 9.1. Namely, we show that the functions

$$\rho_{\alpha\beta}(k, p;\ z_1, z_2;\ k'_\beta) \text{ and } \sigma_{\alpha\beta}(p_\alpha;\ z_1, z_2;\ k'_\beta)$$

may be considered to be the components of the elements $\omega_\beta(z_1, z_2;\ k'_\beta)$, satisfying the equations

$$\omega_\beta(z_1, z_2;\ k'_\beta) = \omega_\beta^{(0)}(z_1, z_2;\ k'_\beta) - \mathbf{A}(z_1)\,\omega_\beta(z_1, z_2;\ k'_\beta), \tag{9.75}$$

where $\omega_\beta^{(0)}(z_1, z_2;\ k'_\beta)$ fulfills the estimate

$$\|\omega_\beta^{(0)}(z_1, z_2;\ k'_\beta)\|_{\theta,\mu} \leqslant C(|z_1 - z_2|), \tag{9.76}$$

and $C(|z_1 - z_2|) \to 0$ for $z_1 \to z_2$.

The elements $\omega_\beta^{(0)}(z_1, z_2;\ k'_\beta)$ must obviously be generated by the integrals

$$w_{\alpha\beta}^{(0)}(k, p;\ z_1, z_2;\ k'_\beta) = (z_2 - z_1)\int \frac{\mathcal{W}_{\alpha\beta}^{(0)}(k, p;\ k', p';\ z_1)}{z_2 + \varkappa_\beta^2 - \frac{p_\beta'^2}{2n_\beta}}\, f_\beta(p'_\beta)\,dp'_\beta,$$

where $\mathcal{W}_{\alpha\beta}^{(0)}(k, p;\ k', p';\ z)$ are kernels of the operators $\mathbf{W}_{\alpha\beta}^{(0)}(z)$ (cf. (3.16)). These kernels have the following representations

$$\mathcal{W}_{\alpha\beta}^{(0)}(k, p;\ k', p';\ z) = \int t_\alpha\left(k_\alpha, k''_\alpha, z - \frac{p_\alpha^2}{2n_\alpha}\right)\frac{\delta(p_\alpha - p''_\alpha)\,\delta(p''_\beta - p'_\beta)}{\frac{k''^2}{2m} + \frac{p''^2}{2n} - z}\, t_\beta\left(k''_\beta, k'_\beta, z - \frac{p_\beta'^2}{2n_\beta}\right)dk''\,dp''.$$

Replacing here $t_\alpha\left(k_\alpha, k'_\alpha, z - \frac{p_\alpha^2}{2n_\alpha}\right)$ by its expression of the type (4.50), we infer that $w_{\alpha\beta}^{(0)}$ naturally falls into two parts

$$w_{\alpha\beta}^{(0)}(k, p;\ z_1, z_2;\ k'_\beta) = \rho_{\alpha\beta}^{(0)}(k, p;\ z_1, z_2;\ k'_\beta) + \frac{\varphi_\alpha(k_\alpha)\,\sigma_{\alpha\beta}^{(0)}(p_\alpha;\ z_1, z_2;\ k'_\beta)}{z_1 + \varkappa_\alpha^2 - \frac{p_\alpha^2}{2n_\alpha}}.$$

We have repeatedly noted before that it is sufficient to consider the functions $\rho_{\alpha\beta}^{(0)}$. The estimates for $\sigma_{\alpha\beta}^{(0)}$ are obtained from the estimates for $\rho_{\alpha\beta}^{(0)}$ by setting $k_\alpha = 0$ in the corresponding estimating functions.

Consider the function $\rho_{\alpha\beta}^{(0)}$, taking to be definite $\alpha = 23$ and $\beta = 31$. We shall omit the indices 23 and 31 in $\rho_{23,\,31}^{(0)}$. Then

$$\rho^{(0)}(k, p;\ z_1, z_2;\ k'_{31}) = (z_2 - z_1)\int \hat{t}_{23}\left(k_{23}, -p'_2 - \frac{m_2}{m_2 + m_3}p_1, z_1 - \frac{p_1^2}{2n_1}\right)\left[\frac{1}{2m_{31}}\left(p_1 + \right.\right.$$
$$\left.\left. + \frac{m_1}{m_3 + m_1}p'_2\right)^2 + \frac{p_2'^2}{2n_2} - z_1\right]^{-1}\left\{\hat{t}_{31}\left(p_1 + \frac{m_1}{m_3 + m_1}p'_2, k'_{31}, z - \frac{p_2'^2}{2n_2}\right) + \right.$$
$$\left. + \frac{\varphi_{31}\left(p_1 + \frac{m_1}{m_3 + m_1}p'_2\right)\overline{\varphi_{31}(k'_{31})}}{z_1 + \varkappa_{31}^2 - \frac{p_2'^2}{2n_2}}\right\}\frac{1}{z_2 + \varkappa_{31}^2 - \frac{p_2'^2}{2n_2}}\, f_2(p'_2)\,dp'_2.$$

We denote by $\rho_1^{(0)}$ the contribution of the first term in the braces to $\rho^{(0)}$. The denominator in the corresponding integrand may be put in the form

$$\left[\frac{1}{\frac{1}{2m_{23}}\left(p_2'+\frac{m_2}{m_2+m_3}p_1\right)^2+\frac{p_1^2}{2n_1}-z_1}+\frac{1}{z_2+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}}\right]\times$$

$$\times\frac{z_2-z_1}{\frac{1}{2m_{31}}\left(p_1+\frac{m_1}{m_3+m_1}p_2'\right)^2+\varkappa_{31}^2+z_2-z_1}.$$

If $|z_1-z_2|\leqslant\frac{1}{2}\varkappa_{31}^2$, then the second factor is analytic in z_1 and z_2 and vanishes for $z_1\to z_2$. It may be shown by the lemma on singular integrals (using the usual estimates of integrals over angle variables) that $\rho^{(0)}(k, p; z_1, z_2; k_{31}')\in\mathfrak{M}(\theta,\mu)$ with some θ, μ for any $z_1\in\Pi_{-\varkappa^2}$, and for z_2 such that $|z_1-z_2|\leqslant\frac{1}{2}\varkappa_{31}^2$, and that $|\rho^{(0)}|\to 0$ for $z_1\to z_2$.

The integrand in the remaining contribution to $\rho^{(0)}$ contains three singular denominators. The product of the first two may be written in the form

$$\left[\frac{1}{\frac{1}{2m_{31}}\left(p_1+\frac{m_1}{m_3+m_1}p_2'\right)^2+\frac{p_2'^2}{2n_2}-z_1}+\frac{1}{z_1+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}}\right]\times$$

$$\times\frac{1}{\frac{1}{2m_{31}}\left(p_1+\frac{m_1}{m_3+m_1}p_2'\right)^2+\varkappa_{31}^2}.$$

The integral $\rho_2^{(0)}$ of the first term of this expression is treated exactly as $\rho_1^{(0)}$. Consider the integral of the second term

$$\rho_3^{(0)}=(z_2-z_1)\int\hat{t}_{23}\left(k_{23},\ -p_2'-\frac{m_2}{m_2+m_3}p_1,\ z_1-\frac{p_1^2}{2n_1}\right)\times$$

$$\times\psi_{31}\left(p_1+\frac{m_1}{m_3+m_1}p_2'\right)\overline{\varphi_{31}(k_{31}')}\left(z_1+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}\right)^{-1}\times$$

$$\times\left(z_2+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}\right)^{-1}f_2(p_2')\,dp_2'.$$

The denominators may be rewritten

$$\frac{z_2-z_1}{\left(z_1+\varkappa_{31}^2-\frac{p_2'^2}{2n_1}\right)\left(z_2+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}\right)}=\frac{1}{z_1+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}}-\frac{1}{z_2+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}},$$

so that $\rho_3^{(0)}(k, p; z_1, z_2; k_{31}')$ is represented as the difference

$$\rho_3^{(0)}(k, p;\ z_1,\ z_2;\ k_{31}')=[\rho(k, p;\ z_1,\ z_1;\ k_{31}')-\rho(k, p;\ z_1,\ z_2;\ k_{31}')]\,\varphi_{31}(k_{31}')$$

of integrals of the type

$$\rho(k, p;\ z_1,\ z_2;\ k_{31}')=$$

$$=\int\hat{t}_{23}\left(k_{23}, -p_2'-\frac{m_2}{m_2+m_3}p_1,\ z_1-\frac{p_1^2}{2n_1}\right)\frac{\psi_{31}\left(p_1+\frac{m_1}{m_3+m_1}p_2'\right)}{z_2+\varkappa_{31}^2-\frac{p_2'^2}{2n_2}}f_2(p_2')\,dp_2'.$$

We then obtain by the lemma on singular integrals the following estimates for $\rho(k, p; z_1, z_2; k_{31}')$

$$|\rho(k, p;\ z_1,\ z_2;\ k_{31}')|\leqslant CN(k, p;\ \theta),$$

$$|\rho(k+h,\ p+l,\ z_1+\Delta_1,\ z_2+\Delta_2;\ k_{31}')-\rho(k, p;\ z_1,\ z_2;\ k_{31}')|\leqslant$$

$$\leqslant CN(k, p;\ \theta)[|h|^{\mu}+|l|^{\mu}+|\Delta_1|^{\mu}+|\Delta_2|^{\mu}],$$

which imply that $\rho_3^{(0)}(k, p;\ z_1, z_2;\ k'_{31})$ also satisfies similar estimates but with a constant $C(|z_1 - z_2|)$ which vanishes for $z_1 \to z_2$.

The functions $\rho_{\alpha\beta}^{(0)}(k, p;\ z_1, z_2;\ k'_\beta)$ with any indices α and β are treated similarly. We conclude that the functions $\rho_{\alpha\beta}^{(0)}(k, p;\ z_1, z_2;\ k'_\beta)$ and $\sigma_{\alpha\beta}^{(0)}(p_\alpha;\ z_1, z_2;\ k'_\beta)$ can be regarded as the components of the elements $\omega_\beta^{(0)}(z_1, z_2;\ k'_\beta)$, which satisfy the estimates (9.76).

By definition of the functions $\rho_{\alpha\beta}^{(0)}(k, p;\ z_1, z_2;\ k'_\beta)$, $\sigma_{\alpha\beta}^{(0)}(p_\alpha;\ z_1, z_2;\ k'_\beta)$, the components of the solutions of equations (9.75) coincide with $\rho_{\alpha\beta}(k, p;\ z_1, z_2;\ k'_\beta)$ and $\sigma_{\alpha\beta}(p_\alpha;\ z_1, z_2;\ k'_\beta)$. Then we may infer from Theorem 7.2 that

$$\|\omega_\beta(z_1, z_2 :\ k'_\beta)\| \leqslant C(|z_1 - z_2|), \tag{9.77}$$

where $C(|z_1 - z_2|) \to 0$ for $z_1 \to z_2$, for any z_1 in a finite region of $\Pi_{-\varkappa^2}$ that contains no singular points, and such z_2 as satisfy $|z_2 - z_1| \leqslant \delta$, where $\delta \leqslant \frac{1}{2} \min\limits_\alpha \varkappa_\alpha^2$.

The estimates for $\rho_{\alpha\beta}(k, p;\ z_1, z_2;\ k'_\beta)$ and $\sigma_{\alpha\beta}(p_\alpha;\ z_1, z_2;\ k'_\beta)$ that are stated in the lemma follow from (9.77). This completes the proof.

The proofs of Lemmas 9.4 and 9.6 follow from Lemma 9.12 and the preceding arguments.

§ 10. Foundation of the time-dependent formulation of the scattering problem for a system described by the operator $\mathbf{h}$

In this section we shall show that the operators $\mathbf{u}^{(+)}$ and $\mathbf{u}^{(-)}$, introduced in § 8, play an important part in the description of the asymptotic behavior for $t \to \pm\infty$ of the solution of the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t} f(t) = \mathbf{h} f(t). \tag{10.1}$$

Theorem 10.1. The operator

$$\mathbf{u}(t) = \exp\{i\mathbf{h}t\} \exp\{-i\mathbf{h}_0 t\} \tag{10.2}$$

strongly converges for $t \to \pm\infty$, *and*

$$\lim_{t \to \pm\infty} \mathbf{u}(t) = \mathbf{u}^{(\pm)}. \tag{10.3}$$

The proof is given below.

Let us remark that a solution of equation (10.1) that satisfies the initial condition $f(t)|_{t=0} = f^{(0)}$ may be written in the form

$$f(t) = \exp\{-i\mathbf{h}t\} f^{(0)},$$

so that Theorem 10.1 bears on the asymptotic behavior of the solution of equation (10.1) for $t \to \pm\infty$. It may be shown by means of Theorem 10.1 that the following formulation of the time-dependent scattering problem is meaningful for a system with the energy operator $\mathbf{h}$:

It is required to find a solution of equation (10.1) which satisfies the condition

$$\lim_{t \to -\infty} \|f(t) - \exp\{-i\mathbf{h}_0 t\} f_-\| = 0, \tag{10.4}$$

where f_- is a given element, and to prove that there exists an element f_+, such that this solution has the following asymptotic behavior for $t \to +\infty$

$$\lim_{t \to +\infty} \| f(t) - \exp\{-i\mathbf{h}_0 t\} f_+ \| = 0. \tag{10.5}$$

Theorem 10.2. The problem just stated has a unique solution, and

$$f_+ = \mathbf{s} f_-, \tag{10.6}$$

where $\mathbf{s}$ *is the unitary operator defined in § 8.*

Theorem 10.2 gives a mathematical justification for the time-dependent formulation of the scattering problem for the considered system. The element $\{-i\mathbf{h}_0 t\} f_-$ describes for $t \to -\infty$ the free relative motion of two far removed and noninteracting particles. The solution $f(t)$ describes the behavior of the system during the collision between these particles. The asymptotic condition (10.5) shows that after the collision the particles are again widely separated and the motion is free. All the information about any change in this free motion as a result of the interaction is contained in the operator $\mathbf{s}$. This operator is called the scattering operator.

We now proceed to prove the theorems enunciated above. We start by showing that Theorem 10.2 is a consequence of Theorem 10.1. Consider the element

$$f(t) = \exp\{-i\mathbf{h}t\} \mathbf{u}^{(-)} f_-. \tag{10.7}$$

We shall show that it solves the stated problem. In fact, we have, in virtue of the unitarity of $\exp\{-i\mathbf{h}t\}$,

$$\| f(t) - \exp\{-i\mathbf{h}_0 t\} f_- \| = \| [\mathbf{u}^{(-)} - \exp\{i\mathbf{h}t\} \exp\{-i\mathbf{h}_0 t\}] f_- \|,$$

and by Theorem 10.1 the right-hand side vanishes for $t \to -\infty$. This solution is unique. To see this, suppose that $f_1(t)$ and $f_2(t)$ are two distinct solutions. We write $f_1^{(0)} = f_1(0)$ and $f_2^{(0)} = f_2(0)$. Then

$$\| f_1^{(0)} - f_2^{(0)} \| = \| f_1(t) - f_2(t) \| \leqslant \| f_1(t) - \exp\{-i\mathbf{h}_0 t\} f_- \| + \\ + \| f_2(t) - \exp\{-i\mathbf{h}_0 t\} f_- \|,$$

and the right-hand side may be made arbitrarily small for $t \to -\infty$.

Let us now prove the asymptotic relation (10.5). Writing

$$f^{(0)} = f(t)|_{t=0} = \mathbf{u}^{(-)} f_-.$$

we have, in view of (8.34),

$$f^{(0)} = \mathbf{u}^{(+)} \mathbf{s} f_- = \mathbf{u}^{(+)} f_+,$$

where $f_+ = \mathbf{s} f_-$, and thus

$$f(t) = \exp\{-i\mathbf{h}t\} \mathbf{u}^{(+)} f_+.$$

Repeating the above arguments we may now show that (10.5) is satisfied for our choice of f_+. This completes the proof.

In order to prove Theorem 10.1 it is sufficient to prove the following

Lemma 10.1. Let $f,\ f' \in \mathfrak{d}_0$. *Then*

$$\lim_{t \to \pm\infty} (\exp\{-i\mathbf{h}t\} \mathbf{u}^{(\pm)} f,\ \exp\{-i\mathbf{h}_0 t\} f') = (f,\ f'). \tag{10.8}$$

We shall show that this lemma entails Theorem 10.1. Since the operators $\mathbf{u}^{(+)}$ and $\mathbf{u}^{(-)}$ are isometric

$$\| \exp\{i\mathbf{h}t\} \exp\{-i\mathbf{h}_0 t\} f - \mathbf{u}^{(\pm)} f \|^2 = 2\{(f,\ f) - \mathrm{Re}\,(\mathbf{u}^{(\pm)} f,\ \exp\{i\mathbf{h}t\} \exp\{-i\mathbf{h}_0 t\} f)\},$$

and here the right-hand side vanishes for $t \to \pm\infty$ if our Lemma 10.1 is valid. Thus, the operator $\mathbf{u}(t)$ converges on the elements $f \in \mathfrak{h}_0$. We conclude, in view of the denseness of the set $\mathfrak{h}_0$ in $\mathfrak{h}$ and the unitarity of the operator $\mathbf{u}(t)$, that $\mathbf{u}(t)$ converges strongly, which proves Theorem 10.1.

We still have to prove Lemma 10.1. Applying (8.15), we rewrite the scalar product on the left-hand side of (10.8)

$$\begin{aligned}(\exp\{-i\mathbf{h}t\}\,\mathbf{u}^{(\pm)}f,\ \exp\{-i\mathbf{h}_0t\}f') &= \\ =(\mathbf{u}^{(\pm)}\exp\{-i\mathbf{h}_0t\}f,\ \exp\{-i\mathbf{h}_0t\}f') &= \\ =(f,\ f')-(\mathbf{k}(\mp 0,\ \mp 0)\exp\{-i\mathbf{h}_0t\}f,\ \exp\{-i\mathbf{h}_0t\}f'). & \qquad (10.9)\end{aligned}$$

It remains to show that the second term in (10.9) vanishes in the limit for $t \to \pm\infty$. Let us write it out as

$$(\mathbf{k}(\mp 0,\ \mp 0)\exp\{-i\mathbf{h}_0t\}f,\ \exp\{-i\mathbf{h}_0t\}f' =$$

$$= \lim_{\varepsilon\to\mp 0}\int \overline{f'(k)}\,\frac{t\left(k,\ k',\ \frac{k'^2}{2m}+i\varepsilon\right)}{\frac{k^2}{2m}-\frac{k'^2}{2m}-i\varepsilon}\exp\left\{i\left[\frac{k^2}{2m}-\frac{k'^2}{2m}\right]t\right\}f(k')\,dk\,dk'.$$

Upon integration over the angle variables and the substitutions $\frac{k^2}{2m}=x$, $\frac{k'^2}{2m}=y$, this integral reduces to

$$I(t,\ \epsilon)=\int_0^A dx\int_0^A dy\,\Phi(x,\ y)\,\frac{\exp\{i(x-y)t\}}{x-y-i\varepsilon}.$$

Here A depends on the radius of the sphere in the exterior of which $f(k)=f'(k)=0$. We now invoke the well-known proposition:

Lemma 10.2. Let $\Phi(x,\ y)$ *be a bounded Hölder function, defined in the square* $0\leqslant x\leqslant A$, $0\leqslant y\leqslant A$. *Then*

$$\lim_{t\to\pm\infty}\lim_{\varepsilon\to\mp 0} I(t,\ \varepsilon)=0.$$

This is often expressed symbolically as

$$\frac{\exp\{itx\}}{x\pm i0}\to 0;\ t\to\pm\infty.$$

The proof of Lemma 10.1 follows from the preceding arguments and Lemma 10.2.

§ 11. On the time-dependent formulation of the scattering problem for a system described by the operator $\mathbf{H}$

In this section we demonstrate the relation between the operators $\mathbf{U}^{(\pm)}$, defined in § 9, and the asymptotic solutions of the Schrödinger equation

$$i\frac{\partial f(t)}{\partial t}=\mathbf{H}f(t) \qquad (11.1)$$

for $|t|\to\infty$.

The operators $\mathbf{U}_0^{(\pm)}$ and $\mathbf{U}_\alpha^{(\pm)}$, $\alpha=23,\ 31,\ 12$ which were used to construct $\mathbf{U}^{(\pm)}$, map $\mathfrak{H}_0$ and $\mathfrak{H}_\alpha$ into $\mathfrak{H}$. We shall now construct from these operators the closely related operators $\tilde{\mathbf{U}}_0^{(\pm)}$ and $\tilde{\mathbf{U}}_\alpha^{(\pm)}$, which operate within $\mathfrak{H}$. To this end we introduce a few definitions.

We denote by $\mathbf{I}_0$ the identity operator that maps $\mathfrak{H}$ into $\mathfrak{H}_0$, i. e. , the inverse of the $\hat{\mathbf{E}}_0$ of § 9. Let $\mathbf{P}_\alpha$, $\alpha=23,\ 31,\ 12$, be the projection operator in

$\mathfrak{H}$ on the subspace spanned by the functions

$$f_\alpha(k, p)=\psi_\alpha(k_\alpha)f_\alpha(p_\alpha), \tag{11.2}$$

where $f_\alpha(p)$ is an arbitrary square-integrable function. Finally, let $\mathbf{I}_\alpha$ be the operator which maps $\mathbf{P}_\alpha\mathfrak{H}$ into $\mathfrak{H}_\alpha$, assigning to a function of the type (11.2) the element $\mathfrak{f}_\alpha \in \mathfrak{H}_\alpha$, represented by the function $f(p_\alpha)$.

Our new operators $\tilde{\mathbf{U}}_0^{(\pm)}$ and $\tilde{\mathbf{U}}_\alpha^{(\pm)}$, $\alpha=23, 31, 12$ are defined as follows

$$\tilde{\mathbf{U}}_0^{(\pm)}=\mathbf{U}_0^{(\pm)}\mathbf{I}_0; \tag{11.3}$$

$$\tilde{\mathbf{U}}_\alpha^{(\pm)}=\mathbf{U}_\alpha^{(\pm)}\mathbf{I}_\alpha\mathbf{P}_\alpha, \quad \alpha=23, 31, 12. \tag{11.4}$$

It is clear that the operators $\mathbf{U}_0^{(\pm)}$, $\mathbf{U}_\alpha^{(\pm)}$ and $\tilde{\mathbf{U}}_0^{(\pm)}$, $\tilde{\mathbf{U}}_\alpha^{(\pm)}$ define each other uniquely.

The main result of the present section is:

Theorem 11.1. The operators

$$\mathbf{U}_0(t)=\exp\{i\mathbf{H}t\}\exp\{-i\mathbf{H}_0t\}; \tag{11.5}$$

$$\mathbf{U}_\alpha(t)=\exp\{i\mathbf{H}t\}\exp\{-i\mathbf{H}_\alpha t\}\mathbf{P}_\alpha \tag{11.6}$$

converge strongly for $t\to\pm\infty$, *and*

$$\lim_{t\to\pm\infty}\mathbf{U}_0(t)=\tilde{\mathbf{U}}_0^{(\pm)}, \tag{11.7}$$

$$\lim_{t\to\pm\infty}\mathbf{U}_\alpha(t)=\tilde{\mathbf{U}}_\alpha^{(\pm)}. \tag{11.8}$$

Since a solution of equation (11.1) that satisfies the initial condition $f(t)|_{t=0}=f^{(0)}$ may be written in the form

$$f(t)=\exp\{-i\mathbf{H}t\}f^{(0)}=\mathbf{W}(t)f^{(0)}, \tag{11.9}$$

we see that Theorem 11 1 concerns the asymptotic behavior of such a solution for $t\to\pm\infty$.

Theorem 11.1 is analogous to Theorem 10.1 which deals with the operator $\mathbf{w}(t)=\exp\{-i\mathbf{h}t\}$. However, there is a considerable difference in the results for the operators $\mathbf{w}(t)$ and $\mathbf{W}(t)$. Comparing (10.2) and (11.5), we see that the operators $\mathbf{u}(t)$ and $\mathbf{U}_0(t)$ have a similar structure. The asymptotic operators $\mathbf{u}^{(+)}$ and $\mathbf{u}^{(-)}$ of $\mathbf{u}(t)$ have an important property, which enables us to introduce the unitary scattering operator $\mathbf{s}=\mathbf{u}^{(+)*}\mathbf{u}^{(-)}$, namely, these operators possess a common range. The operators $\tilde{\mathbf{U}}_0^{(\pm)}$ do not possess this property. Theorem 9.2 shows that only the direct sums of the ranges of $\tilde{\mathbf{U}}_0^{(+)}$, $\tilde{\mathbf{U}}_\alpha^{(+)}$ and of $\tilde{\mathbf{U}}_0^{(-)}$, $\tilde{\mathbf{U}}_\alpha^{(-)}$, $\alpha=23, 31, 12$ coincide.

This difference stems from the basic difference between a two- and a three-body system, mentioned in the introduction. In the language of general scattering theory, the first system is a single channel system, and the second one a multi-channel system. The results derived in the present section serve to give a rigorous mathematical meaning to the concepts of channel, wave operators and scattering operator for a multi-channel system, on the considered example of a three-body system. Namely, it is natural to refer to the subspaces $\mathfrak{H}_0$ and $\mathfrak{H}_\alpha$ of the space $\mathfrak{B}$ as channels, to the operators $\mathbf{U}^{(\pm)}$ as wave operators, and to the operator $\mathbf{S}$ as the scattering operator. All these concepts have been introduced in § 9.

The proof of Theorem 11.1 is, exactly as in § 10, based on the following lemma.

Lemma 11.1. The following propositions are valid:

1. Let $f(k, p)$ and $f'(k, p)$ be finite smooth functions, and let $f(k, p)=0$ in the neighborhood of the singular surfaces $\frac{k^2}{2m}+\frac{p^2}{2n}=\lambda_n$. Then

$$\lim_{t\to\pm\infty}\left(\exp\{-i\mathbf{H}t\}\tilde{\mathbf{U}}_0^{(\pm)}f,\ \exp\{-i\mathbf{H}_0t\}f'\right)=(f, f'). \tag{11.10}$$

2. Let $f(k, p)$ and $f'(k, p)$ have the representation (11.2), i.e., $f\in\mathbf{P}_\alpha\mathfrak{H}$ and $f'\in\mathbf{P}_\alpha\mathfrak{H}$, and let $f_\alpha(p_\alpha)$ and $f'_\alpha(p_\alpha)$ — finite and smooth functions, where $f_\alpha(p_\alpha)=0$ in the neighborhood of the singular surfaces $-\varkappa_\alpha^2+\frac{p_\alpha^2}{2n_\alpha}=\lambda_n$. Then

$$\lim_{t\to\pm\infty}\left(\exp\{-i\mathbf{H}t\}\tilde{\mathbf{U}}_\alpha^{(\pm)}f,\ \exp\{-i\mathbf{H}_\alpha t\}\mathbf{P}_\alpha f'\right)=\left(\mathbf{P}_\alpha f,\ \mathbf{P}_\alpha f'\right). \tag{11.11}$$

Before we go on to prove this lemma let us show that it entails Theorem 11.1. The operators $\mathbf{U}_0^{(\pm)}$ are isometric, and $\mathbf{U}_\alpha^{(\pm)}$, $\alpha=23, 31, 12$ are partly isometric, since $\tilde{\mathbf{U}}_0^{(\pm)}$ and $\tilde{\mathbf{U}}_\alpha^{(\pm)}$, $\alpha=23, 31, 12$ are isometric. More exactly, the following relations hold

$$(\tilde{\mathbf{U}}_0^{(\pm)}f,\ \tilde{\mathbf{U}}_0^{(\pm)}f')=(f, f');$$
$$(\tilde{\mathbf{U}}_\alpha^{(\pm)}f,\ \tilde{\mathbf{U}}_\alpha^{(\pm)}f')=(\mathbf{P}_\alpha f,\ \mathbf{P}_\alpha f').$$

From these relations we obtain

$$\|[\mathbf{U}_0(t)-\tilde{\mathbf{U}}_0^{(\pm)}]f\|^2=2\left[\|f\|^2-\operatorname{Re}\left(\exp\{-i\mathbf{H}t\}\tilde{\mathbf{U}}_0^{(\pm)}f,\ \exp\{-i\mathbf{H}_0t\}f\right)\right];$$
$$\|[\mathbf{U}_\alpha(t)-\tilde{\mathbf{U}}_\alpha^{(\pm)}]f\|^2=2\left[\|\mathbf{P}_\alpha f\|^2-\operatorname{Re}\left(\exp\{-i\mathbf{H}t\}\tilde{\mathbf{U}}_\alpha^{(\pm)}f,\ \exp\{-i\mathbf{H}_\alpha t\}\mathbf{P}_\alpha f\right)\right].$$

By Lemma 11.1 the right-hand members vanish for $t\to\pm\infty$, and we conclude that relations (11.7) and (11.8) are satisfied on dense sets. On account of the boundedness of all the operators involved these relations may be extended throughout the space. Theorem 11.1 then follows.

It remains to prove Lemma 11.1. Comparing the definitions of the operators $\mathbf{H}_\alpha$, $\tilde{\mathbf{H}}_\alpha$ and $\mathbf{L}_\alpha$, $\alpha=0, 23, 31, 12$, we find that

$$\mathbf{H}_0=\mathbf{L}_0^{-1}\tilde{\mathbf{H}}_0\mathbf{L}_0,\quad \mathbf{H}_\alpha\mathbf{P}_\alpha=\mathbf{L}_\alpha^{-1}\tilde{\mathbf{H}}_\alpha\mathbf{L}_\alpha.$$

Applying (9.48) and (9.49) with $\varphi(x)=\exp\{itx\}$, we get the following relations

$$\exp\{i\mathbf{H}t\}\tilde{\mathbf{U}}_0^{(\pm)}=\tilde{\mathbf{U}}_0^{(\pm)}\exp\{i\mathbf{H}_0t\};$$
$$\exp\{i\mathbf{H}t\}\tilde{\mathbf{U}}_\alpha^{(\pm)}=\tilde{\mathbf{U}}_\alpha^{(\pm)}\exp\{i\mathbf{H}_\alpha t\}\mathbf{P}_\alpha,$$

which enable us to write the expressions on the left-hand side of (11.10) and (11.11) in the form

$$(\tilde{\mathbf{U}}_0^{(\pm)}\exp\{-i\mathbf{H}_0t\}f,\ \exp\{-i\mathbf{H}_0t\}f')$$

and

$$(\tilde{\mathbf{U}}_\alpha^{(\pm)}\exp\{-i\mathbf{H}_\alpha t\}\mathbf{P}_\alpha f,\ \exp\{-i\mathbf{H}_\alpha t\}\mathbf{P}_\alpha f')$$

respectively.

It is sufficient for the proof of this lemma to show that

$$I_0^{(\pm)}(t)=([\tilde{\mathbf{U}}_0^{(\pm)}-\mathbf{E}]\exp\{-i\mathbf{H}_0t\}f,\ \exp\{-i\mathbf{H}_0t\}f') \tag{11.12}$$

and

$$I_\alpha^{(\pm)}(t)=([\tilde{\mathbf{U}}_\alpha^{(\pm)}-\mathbf{P}_\alpha]\exp\{-i\mathbf{H}_\alpha t\}\mathbf{P}_\alpha f,\ \exp\{-i\mathbf{H}_\alpha t\}\mathbf{P}_\alpha f') \tag{11.13}$$

vanish for $t\to\pm\infty$, if f and f' have the properties required by the conditions of the lemma. Let us first consider (11.2). By the definition of $\mathbf{U}_0^{(\pm)}$ (cf.

(9.26)) this may be rewritten in the form

$$I_0^{(\pm)}(t) = \lim_{\varepsilon\to\pm 0} \sum I_{\alpha\beta}(t, \varepsilon),$$

where

$$I_{\alpha\beta}(t, \varepsilon) = -\int \overline{f'(k, p)}\,\mathscr{M}_{\alpha\beta}\left(k, p; k', p'; \frac{k'^2}{2m} + \frac{p'^2}{2n} + i\varepsilon\right) f(k', p') \times$$

$$\times \frac{\exp\left\{i\left[\frac{k^2}{2m} + \frac{p^2}{2n} - \frac{k'^2}{2m} - \frac{p'^2}{2n}\right]t\right\}}{\frac{k^2}{2m} + \frac{p^2}{2n} - \frac{k'^2}{2m} - \frac{p'^2}{2n} - i\varepsilon}\, dk\,dp\,dk'\,dp'.$$

Introducing the notation

$$\frac{k^2}{2m} + \frac{p^2}{2n} = x, \quad \frac{k'^2}{2m} + \frac{p'^2}{2n} = y$$

and

$$\Phi_{\alpha\beta}(x, y, \varepsilon) = -\int \delta\left(\frac{k^2}{2m} + \frac{p^2}{2n} - x\right) \overline{f'(k, p)} \times$$

$$\times \mathscr{M}_{\alpha\beta}\left(k, p; k', p'; \frac{k'^2}{2m} + \frac{p'^2}{2n} + i\varepsilon\right) f(k', p')\, \delta\left(\frac{k'^2}{2m} + \frac{p'^2}{2n} - y\right) dk\,dp\,dk'\,dp',$$

we may write $I_{\alpha\beta}(t, \varepsilon)$ in the form

$$I_{\alpha\beta}(t, \varepsilon) = \int_0^A dx \int_0^A dy\, \Phi_{\alpha\beta}(x, y; \varepsilon) \frac{\exp\{i(x - y)t\}}{x - y - i\varepsilon}.$$

One easily verifies that the function

$$m_{\alpha\beta}(k', p'; x, \varepsilon) = \int \mathscr{M}_{\alpha\beta}\left(k, p; k', p'; \frac{k'^2}{2m} + \frac{p'^2}{2n} + i\varepsilon\right) \times$$

$$\times \overline{f'(k, p)}\, \delta\left(\frac{k^2}{2m} + \frac{p^2}{2n} - x\right) dk\,dp$$

encountered in the construction of $\Phi_{\alpha\beta}(x, y; \varepsilon)$, may be expressed as a difference of functions $m_{\alpha\beta}(k, k; z_1, z_2; f')$, (cf. (9.17)), with $z_1 = \frac{k'^2}{2m} + \frac{p'^2}{2n} + i\varepsilon$ and $z_2 = x \pm i0$. Hence this function possesses the properties stated in Lemma 9.1. In particular, it follows from this lemma that $m_{\alpha\beta}(k', p'; x, \varepsilon)$ is a smooth function of all its arguments for any k', p' which lie in a bounded region that does not contain the singular surfaces $\frac{k'^2}{2m} + \frac{p'^2}{2n} = \lambda_n$. Since $f(k', p')$ vanishes in the neighborhood of these surfaces, the product

$$f(k', p')\, m_{\alpha\beta}(k', p', x, \varepsilon)$$

is a smooth bounded function for all values of its arguments, and, consequently, the same goes for $\Phi_{\alpha\beta}(x, y; \varepsilon)$ too. From Lemma 10.2 it now follows that $I_{\alpha\beta}^{(\pm)}(t)$ vanishes for $t \to \pm\infty$, which proves the first statement of Lemma 11.1.

Consider now (11.13), written in the form

$$I_\alpha^{(\pm)}(t) = \lim_{\varepsilon\to\mp 0} \sum_\beta \tilde{I}_{\alpha\beta}(t, \varepsilon),$$

where

$$\tilde{I}_{\alpha\beta}(t, \varepsilon) = -\int \overline{\psi_\alpha(k_\alpha)}\,\overline{f_\alpha'(p_\alpha)}\,\mathscr{K}_{\beta\alpha}\left(k, p; p_\alpha'; -\varkappa_\alpha^2 + \frac{p_\alpha'^2}{2n_\alpha} + i\varepsilon\right) f_\alpha(p_\alpha') \times$$

$$\times \frac{\exp\left\{i\left[\frac{p_\alpha^2}{2n_\alpha} - \frac{p_\alpha'^2}{2n_\alpha}\right]t\right\}}{\frac{k^2}{2m} + \frac{p^2}{2n} + \varkappa_\alpha^2 - \frac{p_\alpha'^2}{2n_\alpha} - i\varepsilon}\, dk\,dp\,dp_\alpha.$$

Let us express $\mathscr{K}_{\beta\alpha}$ in terms of the components $\mathscr{G}_{\beta\alpha}$ and $\mathscr{H}_{\beta\alpha}$ according to a formula of the type (5.23). We obtain

$$\tilde{I}_{\alpha\beta}(t,\varepsilon) = -\int \overline{\psi_\alpha(k_\alpha)}\,\overline{f'_\alpha(p_\alpha)}\,f_\alpha(p'_\alpha)\,\frac{\exp\left\{i\left[\frac{p_\alpha^2}{2n_\alpha}-\frac{p_\alpha'^2}{2n_\alpha}\right]t\right\}}{\frac{k^2}{2m}+\frac{p^2}{2n}+\varkappa_\alpha^2-\frac{p_\alpha'^2}{2n_\alpha}-i\varepsilon}\left[\mathscr{G}_{\alpha\beta}\left(k, p; p'_\alpha; -\varkappa_\alpha^2+\frac{p_\alpha'^2}{2n_\alpha}+i\varepsilon\right)+\right.$$

$$\left.+\frac{\varphi_\beta(k_\beta)\mathscr{H}_{\beta\alpha}\left(p_\beta;\, p'_\alpha;\, -\alpha_\alpha^2+\frac{p_\alpha'^2}{2n_\alpha}+i\varepsilon\right)}{-\varkappa_\alpha^2+\frac{p_\alpha'^2}{2n_\alpha}+\varkappa_\beta^2-\frac{p_\beta^2}{2n_\beta}+i\varepsilon}\right]dk\,dp\,dp_\alpha.$$

The product of the singular denominators may be written as

$$\left[\left(\frac{k^2}{2m}+\frac{p^2}{2n}+\varkappa_\alpha^2-\frac{p_\alpha'^2}{2n_\alpha}-i\varepsilon\right)^{-1}+\right.$$

$$\left.+\left(-\varkappa_\alpha^2+\frac{p_\alpha'^2}{2n_\alpha}+\varkappa_\beta^2-\frac{p_\beta^2}{2n_\beta}+i\varepsilon\right)^{-1}\right]\left(\frac{k_\beta^2}{2m_\beta}+\varkappa_\beta^2\right)^{-1}.$$

The functions in the numerator of the integrand in $\tilde{I}_{\alpha\beta}(t,\varepsilon)$ may be assumed to be smooth. Though the components of the kernels $\mathscr{W}_{\alpha\beta}$ are singular when $z=\lambda_n$, these singularities disappear since the function $f_\alpha(p'_\alpha)$ vanishes by definition in the neighborhood of the surfaces $-\varkappa_\alpha^2+\frac{p_\alpha'^2}{2n_\alpha}=\lambda_n$. The contribution of $\mathscr{G}_{\beta\alpha}$ and $\mathscr{H}_{\beta\alpha}$ from the first iterations of the system (3.20) cannot destroy the smoothness, since by Lemma 6.5 the corresponding kernels admit no secondary singularities for z values in the neighborhood of $-\varkappa_\alpha^2+\frac{p_\alpha'^2}{2n_\alpha}$.

If $\alpha\neq\beta$, we may take p_α and p_β as the integration variables in the estimation of $\tilde{I}_{\alpha\beta}(t,\varepsilon)$. In integrating with respect to p_β we have an ordinary singular integral. After this integration $\tilde{I}_{\alpha\beta}$ becomes

$$\tilde{I}_{\alpha\beta}(t,\varepsilon)=\int\Phi(p_\alpha, p'_\alpha;\varepsilon)\exp\left\{i\left[\frac{p_\alpha^2}{2n_\alpha}-\frac{p_\alpha'^2}{2n_\alpha}\right]t\right\}dp_\alpha\,dp'_\alpha,$$

where $\Phi(p_\alpha, p'_\alpha;\varepsilon)$ is finite, smooth and uniformly bounded in ε. Now by the Riemann-Lebesgue lemma $\tilde{I}_{\alpha\beta}(t,\varepsilon)$, $\alpha\neq\beta$, tends to zero uniformly in ε as $|t|\to\infty$.

If $\alpha=\beta$, we take k_α and p_α as integration variables. The term with the denominator $\left[\frac{k^2}{2m}+\frac{p^2}{2n}+\varkappa_\alpha^2-\frac{p_\alpha'^2}{2n_\alpha}-i\varepsilon\right]^{-1}$ becomes upon integration with respect to k_α a smooth function of p_α and p'_α, and its contribution to $\tilde{I}_{\alpha\alpha}$ vanishes for $|t|\to\infty$. The remaining term is simply the normalization integral for the function $\psi_\alpha(k_\alpha)$. We finally obtain for $|t|\to\infty$

$$I_\alpha^{(\pm)}(t)=\lim_{\varepsilon\to\mp0}\int\overline{f'_\alpha(p_\alpha)}\,\mathscr{H}_{\alpha\alpha}\left(p_\alpha, p'_\alpha, -\varkappa_\alpha^2+\frac{p_\alpha'^2}{2n_\alpha}+i\varepsilon\right)f_\alpha(p'_\alpha)\times$$

$$\times\exp\left\{i\left[\frac{p_\alpha^2}{2n_\alpha}-\frac{p'^2}{2n_\alpha}\right]t\right\}\left[\frac{p_\alpha^2}{2n_\alpha}-\frac{p_\alpha'^2}{2n_\alpha}-i\varepsilon\right]^{-1}dp_\alpha\,dp'_\alpha+o(1).$$

By Lemma 10.2 the first term vanishes for $t\to\pm\infty$, which confirms the second assertion of our lemma and completes the proof.

APPENDIX I

Properties of Hölder functions

In this appendix we prove the propositions concerning Hölder functions stated in § 4. These statements refer to functions $f(k_1, \ldots, k_n, z_1, \ldots, z_m)$ of n three-dimensional and m complex variables. Here it is convenient to designate collectively all, or almost all, of these variables by a single letter and refer to functions $f(x)$, defined on a metric space $\mathfrak{X}$ and having values in the metric space $\mathfrak{F}$. We denote by $|x - x'|$ and $|f(x) - f(x')|$ the distance in $\mathfrak{X}$ and $\mathfrak{F}$, respectively.

Lemma I.1. Let $f(x_1, x_2)$ *be a function defined on* $\mathfrak{X} \otimes \mathfrak{X}$ *with values in* $\mathfrak{J}$, *and*

$$\left|f(x_1, x_2) - f(x_1', x_2')\right| \leqslant C\left(\left|x_1 - x_1'\right|^{\mu_1} + \left|x_2 - x_2'\right|^{\mu_2}\right). \qquad (I.1)$$

Then

$$f(x) = f(x, x)$$

satisfies the condition

$$|f(x) - f(x')| \leqslant C|x - x'|^{\mu}, \quad |x - x'| \leqslant 1, \qquad (I.2)$$

where $\mu = \min(\mu_1, \mu_2)$.

The proof follows immediately from the following sequence of estimates

$$|f(x) - f(x')| = |f(x, x) - f(x', x')| \leqslant |f(x, x') - f(x', x')| +$$
$$+ |f(x, x) - f(x, x')| \leqslant C\left(|x - x'|^{\mu_1} + |x - x'|^{\mu_2}\right) \leqslant C|x - x'|^{\mu}.$$

Lemma I.2. Let $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be two, generally different metric spaces, and let* $f(x, y)$ *be a function defined on* $\mathfrak{X} \otimes \mathfrak{Y}$ *with values in* $\mathfrak{F}$, *which satisfies the condition*

$$|f(x, y) - f(x', y')| \leqslant C(|x - x'|^{\mu} + |y - y'|^{\nu}). \qquad (I.3)$$

The following estimate is then valid

$$|f(x, y) - f(x, y') - f(x' y) + f(x', y')| \leqslant C|x - x'|^{\mu\gamma}|y - y'|^{\nu(1-\gamma)},$$
$$0 \leqslant \gamma \leqslant 1. \qquad (I.4)$$

Proof. Let

$$|x - x'|^{\mu} \geqslant |y - y'|^{\nu}.$$

Then

$$|f(x, y) - f(x, y') - f(x', y) + f(x', y')| \leqslant |f(x, y) - f(x, y')| +$$
$$+ |f(x', y) - f(x', y')| \leqslant C|y - y'|^{\nu} \leqslant C|y - y'|^{\nu[\gamma + (1-\gamma)]} \leqslant$$
$$\leqslant C|x - x'|^{\mu\gamma}|y - y'|^{\nu(1-\gamma)}.$$

The case $|x - x'|^{\mu} \leqslant |y - y'|^{\nu}$ may be treated similarly, if on the left-hand side of (I.4) the first term is grouped together with the third, and the second with the fourth. This completes the proof.

Propositions I and II of § 4 follow from the preceding lemmas. We now pass to the proof of proposition III, i. e., the lemma on singular integrals. While the required result could not be located in the literature on singular integrals, it is not difficult to reduce it to known results.

One usually considers singular integrals of the type

$$\Phi(z) = \oint_{\gamma} \frac{\varphi(t)}{t-z}\,dt, \tag{I.5}$$

where the contour γ lies in a finite region of the complex plane. In order to assure that integral (I.5) has a meaning for any z up to the contour, we impose the Hölder conditions on $\varphi(t)$

$$|\varphi(t)| \leqslant C; \quad |\varphi(t) - \varphi(t')| \leqslant C\,|t-t'|^{\mu}, \tag{I.6}$$

where $|t-t'|$ is the length of arc between the contour points t and t', μ an index, $0 < \mu \leqslant 1$. If condition (I.6) is fulfilled for arbitrarily small $|t-t'|$, it is also fulfilled for any distance between t and t'.

The integration contour is in our case a straight line extending to infinity. Then conditions (I.6) must be modified by the addition of conditions at infinity. We impose the following conditions

$$\begin{gathered} |\varphi(t)| \leqslant C(1+|t|)^{-\theta}; \\ |\varphi(t+l) - \varphi(t)| \leqslant C(1+|t|)^{-\theta}\,|l|^{\mu}. \end{gathered} \tag{I.7}$$

It is assumed here that $\varphi(t)$ is defined on the real axis $-\infty < t < \infty$, that l is a real number, and that θ and μ are such that $0 < \theta < 1$; $0 < \mu < 1$. If (I.7) is fulfilled for arbitrarily small $|l|$, it may considered to be fulfilled for any $|l| \leqslant A$, where A is a fixed finite number. However, the corresponding constant, generally speaking, increases together with A. We shall usually assume (I.7) to be satisfied for $|l| \leqslant 1$. We introduce the definition

$$\|\varphi\|_{\theta,\mu} = \sup_{t,\,|l| \leqslant 1} (1+|t|)^{\theta} \left\{ |\varphi(t)| + \frac{|\varphi(t+l) - \varphi(t)|}{|l|^{\mu}} \right\}. \tag{I.8}$$

The following proposition is needed:

Let $\varphi(t)$ be defined on the real axis $-\infty < t < \infty$, with $\|\varphi\|_{\theta,\mu} < \infty$ and $\varphi(t) \equiv 0$ for $|t| \geqslant A$. Then

$$\Phi(z) = \int_{-\infty}^{\infty} \frac{\varphi(t)}{t-z}\,dt = \int_{-A}^{A} \frac{\varphi(t)}{t-z}\,dt$$

is an analytic function of the complex variable z throughout the plane slit along the real axis from $-A$ to A, and the following estimates are valid for any z in this plane:

$$\begin{gathered} |\Phi(z)| \leqslant C\,\|\varphi\|_{0,\mu}(1+|z|)^{-1}, \\ |\Phi(z+\Delta) - \Phi(z)| \leqslant C\,\|\varphi\|_{0,\mu}(1+|z|)^{-1}\,|\Delta|^{\mu}, \end{gathered}$$

where C depends only on A and μ.

These estimates are obvious for $|z| \geqslant 2A$, since for such $|z|$ the integral is nonsingular. The proposition follows for small $|z|$ from Privalov's known lemma on the Cauchy principal values of integrals with a Hölder density, and from the maximum principle for analytic functions. The proof of Privalov's lemma and other properties of singular integrals needed for the proof may be found, e. g., in /11/. We will refer to this proposition as Privalov's lemma.

We now apply Privalov's lemma to prove

Lemma I.3. Let $\varphi(t)$ *be defined on the real axis, and let* $\|\varphi(t)\|_{\theta,\mu} < \infty$. *Then the function*

$$\Phi(z) = \int_{-\infty}^{\infty} \frac{\varphi(t)}{t-z}\, dt \tag{I.9}$$

is analytic in the upper and lower half-planes, continuous up to the real axis, and the following estimates are valid for any z *with* $\operatorname{Im} z \geqslant 0$ *or* $\operatorname{Im} z \leqslant 0$:

$$\begin{gathered} |\Phi(z)| \leqslant C\|\varphi\|_{\theta,\mu}(1+|z|)^{-\theta'}, \\ |\Phi(z+\Delta)-\Phi(z)| \leqslant C\|\varphi\|_{\theta,\mu}(1+|z|)^{-\theta'}|\Delta|^{\mu}, \end{gathered} \tag{I.10}$$

where θ' *may be taken arbitrarily close to* θ *from below.*

Proof. We use the usual notations

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z, \quad \rho = |z|, \quad \varphi = \arg z,$$

so that

$$z = x + iy = \rho \exp\{i\varphi\} = \rho\cos\varphi + i\rho\sin\varphi.$$

Consider, say, the case $y \geqslant 0$, and divide the upper half-plane $\Pi^{(+)}$ into strips $\Pi_n^{(+)}$, defined by

$$|x-n| \leqslant \frac{1}{2},$$

where n assumes positive and negative integral values

$$n = \ldots, -2, -1, 0, 1, 2, \ldots$$

Let $\eta(t)$ be the cutoff function

$$\eta(t) = \begin{cases} 1 & |t| \leqslant 1, \\ 0 & |t| \geqslant 2, \end{cases}$$

$$0 \leqslant \eta(t) \leqslant 1; \quad |\eta'(t)| \leqslant C.$$

Consider the case $z \in \Pi_n^{(+)}$ for some fixed n. We express $\Phi(z)$ as

$$\Phi(z) = \int_{-\infty}^{\infty} \frac{\eta(t-n)f(t)}{t-z}\, dt + \int_{-\infty}^{\infty} \frac{[1-\eta(t-n)]f(t)}{t-z}\, dt = \Phi_1^{(n)}(z) + \Phi_2^{(n)}(z)$$

and estimate each term separately. The denominator in $\Phi_2^{(n)}(z)$ is nonsingular, so that $\Phi_2^{(n)}(z)$ is analytic in $\Pi_n^{(+)}$, and

$$\left|\Phi_2^{(n)}(z)\right| \leqslant C\|\varphi\|_{\theta,\mu}\left(\int_{-\infty}^{n-1} + \int_{n+1}^{\infty}\right)(1+|t|)^{-\theta}((t-x)^2+y^2)^{-1/2}\, dt \leqslant$$

$$\leqslant C\|\varphi\|_{\theta,\mu}\int_{1/2}^{\infty}(t-|x|)^{-\theta}(t^2+y^2)^{-1/2}\, dt \leqslant C\|\varphi\|_{\theta,\mu}.$$

If $\rho \geqslant 1$, we may change the integration variable in the last integral, writing $t = \rho t'$, and obtain

$$\left|\Phi_2^{(n)}(z)\right| \leqslant C\|\varphi\|_{\theta,\mu}\rho^{-\theta}\int_{\frac{1}{2\rho}}^{\infty} |t-|\cos\varphi||^{-\theta}(t^2+\sin^2\varphi)^{-1/2}\, dt \leqslant C\|\varphi\|_{\theta,\mu}\rho^{-\theta}\ln\rho.$$

In the last step it is necessary to consider separately the case $\sin^2\varphi > \frac{1}{2}$ and $\sin^2\varphi < \frac{1}{2}$.

We finally obtain

$$\left|\Phi_2^{(n)}(z)\right| \leqslant C\|\varphi\|_{\theta,\mu}(1+|z|)^{-\theta'}. \tag{I.11}$$

Similarly

$$\left|\frac{d}{dz}\Phi_2^{(n)}(z)\right| \leqslant C\|\varphi\|_{\theta,\mu}\int\limits_{1/2}^{\infty}|t-|x||^{-\theta}(t^2+y^2)^{-1}\,dt \leqslant C\|\varphi\|_{\theta,\mu}(1+|z|)^{-\theta}. \tag{I.12}$$

Consider now $\Phi_1^{(n)}(z)$. We make in the integral of $\Phi_1^{(n)}(z)$ the substitution

$$t' = t - n$$

and write

$$z' = z - n$$

If $z \in \Pi_n^{(+)}$, then

$$|z'| \geqslant |y|.$$

We obtain

$$\Phi_1^{(n)}(z) = \tilde{\Phi}^{(n)}(z') = \int\limits_{-2}^{2}\frac{\tilde{\varphi}_n(t)}{t-z'}\,dt,$$

where

$$\tilde{\varphi}_n(t) = \eta(t)\,\varphi(t+n).$$

The properties of $\varphi(t)$ imply that

$$|\tilde{\varphi}_n(t)| \leqslant C\|\varphi\|_{\theta,\mu}(1+|n|)^{-\theta},$$
$$|\tilde{\varphi}_n(t+l) - \tilde{\varphi}_n(t)| \leqslant C\|\varphi\|_{\theta,\mu}(1+|n|)^{-\theta}|l|^{\mu}.$$

The following estimates are obtained from Privalov's lemma

$$|\tilde{\Phi}^{(n)}(z')| \leqslant C\|\varphi\|_{\theta,\mu}(1+|n|)^{-\theta}(1+|y|)^{-1}, \tag{I.13}$$
$$|\tilde{\Phi}^{(n)}(z'+\Delta) - \tilde{\Phi}^{(n)}(z')| \leqslant C\|\varphi\|_{\theta,\mu}(1+|n|)^{-\theta}(1+|y|)^{-1}|\Delta|^{\mu}.$$

It is clear that for $z \in \Pi_n^{(+)}$

$$(1+|n|)^{-\theta}(1+|y|)^{-1} \leqslant C(1+|z|)^{-\theta}. \tag{I.14}$$

It follows from (I.13) and (I.14) that $\Phi_1^{(n)}(z)$ satisfies the estimates (2.10) for $z \in \Pi_n^{(+)}$. The same statement for $\Phi_2^{(n)}$ follows from (I.11) and (I.12). This implies, in view of the arbitrariness of n, that the estimates (I.10) are valid for $\Phi(z)$ with any z.

This completes the proof.

Consider now the Hölder function $f(q)$ of the three-dimensional variable q. We assume that $f(q)$ satisfies the conditions

$$|f(q)| \leqslant CM(q),$$
$$|f(q+h) - f(q)| \leqslant CM(q)|h|^{\mu}, \tag{I.15}$$

where

$$M(q) > 0, \quad \int d\Omega_q M(q) \leqslant (1+|q|)^{-1-\theta}.$$

Here h is a three-dimensional vector, $|h| \leqslant 1$, and θ and μ satisfy $0 < \theta < 1$, $0 < \mu < 1$. We write

$$\|f\|_{\theta,\mu} = \sup_{q,\,|h|\leqslant 1} M(q)^{-1}\left\{|f(q)| + \frac{|f(q+h)-f(q)|}{|h|^{\mu}}\right\}. \tag{I.16}$$

Though this notation resembles that of (I.8), misunderstanding is unlikely to occur.

Lemma I.4. Let $m>0$. *The function*

$$\Phi(z)=\int \frac{f(q)}{\frac{q^2}{2m}-z}\,dq \tag{I.17}$$

is analytic in the complex plane Π_0, *slit along the positive real axis, and for any* z *in this plane*

$$|\Phi(z)|\leqslant C\|f\|_{\theta,\mu}(1+|z|)^{-\frac{\theta'}{2}},$$

$$|\Phi(z+\Delta)-\Phi(z)|\leqslant C\|f\|_{\theta,\mu}(1+|z|)^{-\frac{\theta'}{2}}|\Delta|^{\nu}. \tag{I.18}$$

Here $\nu=\min\left(\frac{1}{2},\,\mu\right)$, *and* θ' *may be chosen smaller than* θ *and as close to it as desired.*

For $\operatorname{Re} z<-1$, $\Phi'(z)$ *satisfies the estimate*

$$|\Phi'(z)|\leqslant C\|f\|_{\theta,\mu}(1+|z|)^{-\left(1+\frac{\theta}{2}\right)}. \tag{I.19}$$

Proof. $\Phi(z)$ may be written in spherical coordinates as

$$\Phi(z)=\int_0^{\infty}\frac{|q|\,d|q|}{\frac{q^2}{2m}-z}\bar{\varphi}(|q|), \tag{I.20}$$

where

$$\bar{\varphi}(|q|)=|q|\int d\Omega_q f(q). \tag{I.21}$$

Writing $t=\frac{q^2}{2m}$ and

$$\varphi(t)=\begin{cases} m\bar{\varphi}(\sqrt{2mt}), & t\geqslant 0,\\ 0, & t<0,\end{cases} \tag{I.22}$$

we obtain the representation

$$\Phi(z)=\int_{-\infty}^{\infty}\frac{\varphi(t)}{t-z}\,dt.$$

It may be deduced from (I.15), (I.21), (I.22) that

$$|\varphi(t)|\leqslant C\|f\|_{\theta,\mu}(1+|t|)^{-\frac{\theta}{2}},$$

$$|\varphi(t+l)-\varphi(t)|\leqslant C\|f\|_{\theta,\mu}(1+|t|)^{-\frac{\theta}{2}}|l|^{\nu}, \tag{I.23}$$

where $\nu=\min\left(\mu,\,\frac{1}{2}\right)$. Let us derive the second of these estimates. Let $|l|\leqslant\frac{1}{2}|t|$. If $t<0$, then $t+l<0$ and $\varphi(t)=\varphi(t+l)=0$, so that it is sufficient to consider the case $t>0$. Then

$$|\varphi(t+l)-\varphi(t)|\leqslant C\|f\|_{\theta,\mu}\Big[|\sqrt{t+l}-\sqrt{t}|(1+|t|)^{-\frac{1+\theta}{2}}+$$
$$+\sqrt{t}\,|\sqrt{t+l}-\sqrt{t}|^{\mu}(1+|t|)^{-\frac{1+\theta}{2}}\Big]\leqslant C\|f\|_{\theta,\mu}(1+|t|)^{-\frac{\theta}{2}}\left(|l|^{\frac{1}{2}}+|l|^{\mu}\right).$$

If, however, $|l| \geqslant \frac{1}{2}|t|$, then

$$|\varphi(t+l) - \varphi(t)| \leqslant C\left(|t+l|^{\frac{1}{2}} + |t|^{\frac{1}{2}}\right)\|f\|_{\theta,\mu}(1+|t|)^{-\frac{1+\theta}{2}} \leqslant$$

$$\leqslant C\|f\|_{\theta,\mu}(1+|t|)^{-\frac{\theta}{2}}|l|^{\frac{1}{2}},$$

which completes the proof of the required estimate.

Thus $\varphi(t)$ satisfies the conditions of Lemma I.3, and $\Phi(z)$ is therefore analytic in the upper and lower half-planes and satisfies in each of them the estimates (I.18). It is obvious from (I.20) that $\Phi(z)$ is analytic in the neighborhood of the negative real axis, so that the estimates (I.18) are valid in the plane Π_0.

It remains to prove the estimate (I.19). We readily verify that

$$|\tilde{\varphi}(|q|)| \leqslant C\|f\|_{\theta,\mu}(1+|q|)^{-\theta},$$

whence

$$|\Phi'(z)| \leqslant C\|f\|_{\theta,\mu}\int\limits_0^\infty r\,dr\,(1+r)^{-\theta}\left[\left(\frac{r^2}{2m} - x\right)^2 + y^2\right]^{-1}.$$

This gives for $x<0$

$$|\Phi'(z)| \leqslant C\|f\|_{\theta,\mu}|z|^{-\left(1+\frac{\theta}{2}\right)},$$

and thus (I.19) holds for $x<-1$.

This completes the proof.

Proposition III of § 4 follows from Lemmas I.2 and I.4.

APPENDIX II

Estimates for some integrals

We shall derive here estimates for three integrals which appear in the main text.

We encountered in § 4 the integral

$$I(k, |q|, \theta) = \int d\Omega_q (1+|k-q|)^{-\theta}. \tag{II.1}$$

Lemma II.1. Let $\theta<2$. *Then the following estimate holds for* $I(k, |q|, \theta)$

$$|I(k, |q|, \theta)| \leqslant C(1+|k|)^{-\theta_1}(1+|q|)^{-\theta_2}, \tag{II.2}$$
$$\theta_1 + \theta_2 = \theta.$$

Proof. In spherical coordinates the integral $I(k, |q|, \theta)$ assumes the form

$$I(k, |q|, \theta) = 2\pi\int\limits_{-1}^{1} d\eta\,[1+(k^2 - 2|k||q|\eta + q^2)^{1/2}]^{-\theta},$$

which is symmetric in $|k|$ and $|q|$. Consider the two cases

1. $|k| \leqslant 1$. Then

$$|I(k, |q|, \theta)| \leqslant 2\pi\int\limits_{-1}^{1} d\eta \leqslant 4\pi. \tag{II.3}$$

2. $|k| \geqslant 1.$ Then

$$|I(k, |q|, \theta)| \leqslant 2\pi \int_{-1}^{1} d\eta \, (k^2 - 2|k||q|\eta + q^2)^{-\frac{\theta}{2}} =$$

$$= \frac{2\pi}{2-\theta} \frac{(|k| + |q|)^{2-\theta} - ||k| - |q||^{2-\theta}}{|k||q|} = \frac{1}{|k|^{\theta}} \frac{2\pi}{2-\theta} \frac{(1+t)^{2-\theta} - |1-t|^{2-\theta}}{t},$$

where $t = \frac{|q|}{|k|}$. One easily verifies that if $0 \leqslant \alpha \leqslant 2$, then

$$f(t) = \frac{(1+t)^{\alpha} - |1-t|^{\alpha}}{t}$$

is uniformly bounded for $0 \leqslant t \leqslant \infty$, which implies

$$|I(k, |q|, \theta)| \leqslant C|k|^{-\theta} \tag{II.4}$$

for $\theta < 2$. By (II.3) in conjunction with (II.4) we obtain

$$|I(k, |q|, \theta)| \leqslant C(1 + |k|)^{-\theta}.$$

Now (II.2) follows in view of the symmetry of I with respect to $|k|$ and $|q|$. This completes the proof.

Also in § 4 we had the integral

$$I(a) = \int (1 + |q|)^{-\alpha} (1 + |q - a|)^{-\beta} dq. \tag{II.5}$$

Lemma II.2. Let $\alpha < 3$, $\beta < 3$, *and* $\alpha + \beta > 3$. *Then*

$$|I(a)| \leqslant C(1 + |a|)^{-(\alpha+\beta-3)}. \tag{II.6}$$

Proof. We consider the two cases

1. $|a| \leqslant 1.$ Then

$$|I(a)| \leqslant C \int (1 + |q|)^{-(\alpha+\beta)\lambda} dq \leqslant C, \tag{II.7}$$

since $\alpha + \beta > 3$ by condition.

2. $|a| > 1.$ Then

$$|I(a)| \leqslant \int |q|^{-\alpha} |q - a|^{-\beta} dq \leqslant |a|^{\alpha+\beta-3} \int |q'|^{-\alpha} |q' - \tilde{a}|^{-\beta} dq' \leqslant C|a|^{\alpha+\beta-3}. \tag{II.8}$$

Here $q' = |a| q$, and $\tilde{a} = \frac{a}{|a|}$. Now (II.6) follows from (II.7) and (II.8), which completes the proof.

In § 6 we encountered the integral

$$I(a, b) = \int (1 + q^2)^{-1} (1 + |q - a|)^{-\theta} (1 + |q - b|)^{-\theta} dq. \tag{II.9}$$

Lemma II.3. Let $1 < \theta < 2$. *Then*

$$|I(a, b)| \leqslant C(1 + |a|)^{-\theta}. \tag{II.10}$$

Proof. In spherical coordinates we have

$$|I(a, b)| \leqslant \int_0^{\infty} d|q| \int d\Omega_q (1 + |q - a|)^{-\theta} (1 + |q - b|)^{-\theta}.$$

We take $\delta < 1$, such that $\theta + \delta < 2$, and estimate the integral over the angle variables as follows

$$\int d\Omega_q (1+|q-a|)^{-\theta}(1+|q-b|)^{-\theta} \leqslant \int d\Omega_q (1+|q-a|)^{-\theta}(1+|q-b|)^{-\delta}\times$$

$$\times(1+||q|-|b||)^{-1+\frac{\delta}{2}} \leqslant C(1+|a|)^{-\theta}(1+|q|)^{-\delta}(1+||q|-|b||)^{-1+\frac{\delta}{2}}.$$

Here we relaxed the estimate, replacing $(1+|q-b|)^{-\theta}$ by $(1+|q-b|)^{-\left(1+\frac{\delta}{2}\right)}$, and used the trivial inequality

$$(1+|q-b|)^{-\theta} \leqslant (1+||q|-|b||)^{-\theta}, \quad \theta>0,$$

as well as Hölder's inequality with suitable indices, and Lemma II.1. We finally obtain

$$|I(a, b)| \leqslant C(1+|a|)^{-\theta}\int_0^{\infty} dx x^{-\delta}(1+|x-|b||)^{-1+\frac{\delta}{2}} \leqslant C(1+|a|)^{-\theta}.$$

This completes the proof.

APPENDIX III

Proof of Lemma 6.2

We shall now derive estimates for the integral

$$I(a, b, \xi, \eta)=\int \frac{f(q)\,dq}{[(q-a)^2-\xi][q-b)^2-\eta]}, \tag{III.1}$$

which appeared in Lemma 6.2. Here a, b, q are three-dimensional variables, ξ, η are complex variables, and $f(q)$ a Hölder function which satisfies the conditions

$$|f(q)| \leqslant C; \ |f(q+h)-f(q)| \leqslant C|h|^{\mu} \tag{III.2}$$

and vanishes outside a sphere with radius of the order of $\sqrt{R}$, where $R>1$ is a fixed number. We shall be interested in the explicit dependence of all the estimates on R.

We assume that a, b, ξ, η vary within finite regions of the three-dimensional space and of the complex plane Π_0, and that

$$a^2 \leqslant CR; \ b^2 \leqslant CR; \ |\xi| \leqslant CR; \ |\eta| \leqslant CR. \tag{III.3}$$

It is sufficient to consider the integral (III.1) for $b=0$, i. e.,

$$I(a, \xi, \eta)=\int \frac{f(q)\,dq}{[(q-a)^2-\xi][q^2-\eta]} \tag{III.4}$$

since the general case, for $b\neq 0$, may always be brought to the form

$$I(a, b, \xi, \eta)=\int \frac{f(q+b)\,dq}{[(q-a')^2-\xi]\,[q^2-\eta]} \tag{III.5}$$

by the substitution $q'=q-b$, $a'=a-b$. The smoothness of this integral with respect to the variable b in the numerator follows immediately if we can estimate the integral (III.4) by means of the constants of (III.2) and R. Otherwise the form of (III.5) is exactly as that of (III.4).

The integrand of (III.4) contains two singular denominators, and the two singularities are in general inseparable. It is therefore natural to look for a system of coordinates in which these denominators would depend on different variables. The spherical coordinates around the vector a as axis are suitable for this purpose.

Let us carry out all the steps of this change of variables in (III.4). Let q_1, q_2, q_3 and a_1, a_2, a_3 be the Cartesian components of the vectors q and a in some fixed system with axes O_1, O_2, O_3. It may be assumed without loss of generality that $a_3 \geqslant 0$, otherwise we simply reverse the O_3-axis. We now pass from the integration variables q_1, q_2, q_3 to the new ones r, s, φ, according to the transformation formulas

$$\begin{aligned} q_1 &= a_{11} r\sqrt{1-s^2}\sin\varphi + a_{12} r\sqrt{1-s^2}\cos\varphi + a_{13} rs; \\ q_2 &= a_{21} r\sqrt{1-s^2}\sin\varphi + a_{22} r\sqrt{1-s^2}\cos\varphi + a_{23} rs; \\ q_3 &= a_{31} r\sqrt{1-s^2}\sin\varphi + a_{32} r\sqrt{1-s^2}\cos\varphi + a_{33} rs. \end{aligned} \tag{III.6}$$

Here a_{ij}, $i, j = 1, 2, 3$ are the elements of the orthogonal matrix which represents rotation of the initial rectangular axes around an axis perpendicular to both the O_3-axis and the vector a, so as to align the new O_3-axis with a. The a_{ij} thus depend on the direction cosines of the vector a relative to the old axes, which we denote by $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$. These are the components of the unit vector along a: $\tilde{a} = \frac{a}{|a|}$. The explicit form of the matrix $\| a_{ij}(a) \|$ is

$$\| a_{ij}(a) \| = \begin{pmatrix} \tilde{a}_3 + \frac{\tilde{a}_2^2}{1+\tilde{a}_3} & -\frac{\tilde{a}_1\tilde{a}_2}{1+\tilde{a}_3} & \tilde{a}_1 \\ \frac{\tilde{a}_1\tilde{a}_2}{1+\tilde{a}_3} & \tilde{a}_3 + \frac{\tilde{a}_1^2}{1+\tilde{a}_3} & \tilde{a}_2 \\ -\tilde{a}_1 & -\tilde{a}_2 & \tilde{a}_3 \end{pmatrix}.$$

Note that

$$|\tilde{a}_i| \leqslant 1; \quad |\operatorname{grad} \tilde{a}_i| \leqslant \frac{1}{|a|},$$

whence

$$|a_{ij}(a)| \leqslant 1; \quad |\operatorname{grad} a_{ij}(a)| \leqslant \frac{C}{|a|} \tag{III.7}$$

excluding the neighborhood of the point $\tilde{a}_3 = -1$.

In the new variables r, s, φ the integral (III.4) assumes the form

$$I(a, \xi, \eta) = \int\limits_0^\infty \frac{r^2 dr}{r^2 - \eta} \int\limits_{-1}^{1} ds \frac{\Phi(s, r, a)}{r^2 - 2|a|rs + a^2 - \xi}, \tag{III.8}$$

where

$$\Phi(s, r, a) = \int\limits_0^{2\pi} f(q)\, d\varphi, \tag{III.9}$$

it being understood that the variables q_1, q_2, q_3 in (III.9) are expressed in terms of r, s, φ via (III.6). The integral (III.8) contains two singular denominators, but one of them depends only on r. It is therefore possible first to consider the integral with respect to s as a function of all the variables and then use the obtained results to estimate the integral with respect to r. We must first check the smoothness of the function $\Phi(s, r, a)$.

Lemma III.1. The function $\Phi(s, r, a)$ *is a Hölder function of* r *and* a *for fixed* s, *satisfying*

$$|\Phi(s, r, a)| \leqslant C; \qquad (\text{III.10})_1$$

$$|\Phi(s, r+\Delta, a) - \Phi(s, r, a)| \leqslant C|\Delta|^{\mu}; \qquad (\text{III.10})_2$$

$$|\Phi(s, r, a+h) - \Phi(s, r, a)| \leqslant CR^{\frac{\mu}{2}} \frac{|h|^{\mu}}{|a+\theta h|^{\mu}}. \qquad (\text{III.10})_3$$

Here $a+\theta h$ denotes the point with the coordinates $a_1+\theta_1 h_1$, $a_2+\theta_2 h_2$, $a_3+\theta_3 h_3$, where $\theta_1, \theta_2, \theta_3$ are certain numbers, $0 \leqslant \theta_i \leqslant 1$, $i = 1, 2, 3$.

Proof. The first two estimates follow directly from the properties of the function $f(q)$. To derive the last estimate we consider two cases:

1. $|h| \leqslant \frac{1}{2}|a|$. Then $a_3 + h_3 \geqslant -\frac{1}{2}$, since $a_3 \geqslant 0$, so that we may apply the estimates (III.7) and the mean value theorem in order to verify that

$$|a_{ij}(a+h) - a_{ij}(a)| \leqslant C \frac{|h|}{|a+\theta h|}; \qquad (\text{III.11})$$

2. $|h| \geqslant \frac{1}{2}|a|$. Then

$$|a_{ij}(a+h) - a_{ij}(a)| \leqslant (|a_{ij}(a+h)| + |a_{ij}(a)|) \frac{2|h|}{|a|} \leqslant C\frac{|h|}{|a|},$$

which is a particular case of (III.11) for $\theta_1 = \theta_2 = \theta_3 = 0$. We thus find that for any h

$$|\Phi(s, r, a+h) - \Phi(s, r, a)| \leqslant$$

$$\leqslant Cr^{\mu} \max_{i,j} |a_{ij}(a+h) - a_{ij}(a)|^{\mu} \leqslant CR^{\frac{\mu}{2}} \frac{|h|^{\mu}}{|a+\theta h|^{\mu}}.$$

This proves the lemma.

Lemma III.2. The function $\Phi(s, r, a)$ *is a Hölder function of* s *for fixed* r *and* a, *and*

$$|\Phi(s+\Delta, r, a) - \Phi(s, r, a)| \leqslant Cr^{\mu}|\Delta|^{\frac{\mu}{2}}. \qquad (\text{III.10})_4$$

Proof. Let us first assume that $f(q)$ is a continuously twice-differentiable function

$$|\mathbf{grad}\, f(q)| \leqslant C_1; \quad |\mathbf{grad}^2\, f(q)| \leqslant C_2.$$

We will show that $\Phi(s, r, a)$ is in that case differentiable with respect to s, and

$$\left|\frac{\partial}{\partial s}\Phi(s, r, a)\right| \leqslant C(C_1 r + C_2 r^2). \qquad (\text{III.12})$$

By definition of $\Phi(s, r, a)$ we have

$$\frac{\partial}{\partial s}\Phi(s, r, a) = \int_0^{2\pi} d\varphi \left\{\frac{\partial f(q)}{\partial q_1}\left(a_{11} r \frac{-s}{\sqrt{1-s^2}} \sin\varphi + \right.\right.$$

$$\left.\left. + a_{12} r \frac{-s}{\sqrt{1-s^2}} \cos\varphi + a_{13} r\right) + \frac{\partial f}{\partial q_2}(\ldots) + \frac{\partial f}{\partial q_3}(\ldots)\right\}. \qquad (\text{III.13})$$

The terms indicated by dots are analogous to the first. It follows immediately from (III.13) that

$$\left|\frac{\partial}{\partial s}\Phi(s, r, a)\right| \leqslant CC_1\frac{r}{\sqrt{1-s^2}}. \tag{III.14}$$

The factor $(1-s^2)^{-1/2}$ on the right-hand side of (III.14) may be dropped. Indeed, consider a typical term on the right-hand side of (III.13), containing the singular factor

$$c(s, r, a)=\int\limits_0^{2\pi} d\varphi\,\frac{\partial f(q)}{\partial q_1}\sin\varphi\,\frac{a_{11}rs}{\sqrt{1-s^2}}.$$

The function $\frac{\partial f(q)}{\partial q_1}$ does not depend on φ for $s=\pm 1$ and the integral of $\sin\varphi$ over a complete period vanishes. We now build up this argument into a rigorous proof. To be definite, let us take $s>0$. We then have

$$c(s, r, a)=\int\limits_0^{2\pi} d\varphi\left(\frac{\partial f(q)}{\partial q_1}-\left.\frac{\partial f(q)}{\partial q_1}\right|_{s=1}\right)\sin\varphi\,\frac{a_{11}rs}{\sqrt{1-s^2}}=$$

$$=\int\limits_0^{2\pi} d\varphi\int\limits_s^1 -\frac{\partial^2 f}{\partial q_1\partial s'}\,ds'\sin\varphi\,\frac{a_{11}rs}{\sqrt{1-s^2}}.$$

Repeating the derivation of (III.14), we find the following estimate for the second derivative that appears in the last expression

$$\left|\frac{\partial^2 f}{\partial q_1\partial s}\right| \leqslant CC_2\frac{r}{\sqrt{1-s^2}},$$

which gives

$$|c(s, r, a)| \leqslant CC_2r^2\int\limits_s^1\frac{ds'}{\sqrt{1-s'^2}}\,\frac{1}{\sqrt{1-s^2}} \leqslant CC_2r^2,$$

and hence (III.12) follows.

Let $f(q)$ now satisfy only condition (III.12). In this case we may choose for $f(q)$ a mean-valued $f_h(q)$ such that

$$|f(q)-f_h(q)| \leqslant C|h|^{\mu};$$

$$|\operatorname{grad} f_h(q)| \leqslant \frac{C}{|h|^{1-\mu}};\ |\operatorname{grad}^2 f_h(q)| \leqslant \frac{C}{|h|^{2-\mu}}.$$

We construct the function $\Phi_h(s, r, a)$ from $f_h(q)$ according to (III.9). We now have

$$|\Phi(s+\Delta, r, a)-\Phi(s, r, a)| \leqslant |\Phi(s+\Delta, r, a)-\Phi_h(s+\Delta, r, a)|+$$

$$+|\Phi(s, r, a)-\Phi_h(s, r, a)|+|\Phi_h(s+\Delta, r, a)-\Phi_h(s, r, a)| \leqslant$$

$$\leqslant C\left[|h|^{\mu}+\left(\frac{r}{|h|^{1-\mu}}+\frac{r^2}{|h|^{2-\mu}}\right)\Delta\right].$$

Taking for the averaging parameter $h=r\Delta^{1/2}$, we finally obtain

$$|\Phi(s+\Delta, r, a)-\Phi(s, r, a)| \leqslant Cr^{\mu}|\Delta|^{\frac{\mu}{2}}.$$

This completes the proof.

Having clarified the behavior of the function $\Phi(s, r, a)$, we may turn to the study of the internal integral in (III.8), which has the form

$$I(r, a, \xi)=\int\limits_{-1}^{1}\frac{\Phi(s)\,ds}{r^2-2|a|rs+a^2-\xi}. \tag{III.15}$$

Here and in the following we shall occasionally omit to write the arguments r and a of $\Phi(r, s, a)$. The integral (III.15) is an ordinary singular integral over a finite interval, and has therefore known logarithmic singularities. In order to separate these singularities, consider a smooth interpolation function $\varphi(s)$ having the properties

$$\varphi(s)=\begin{cases} 1 & -1\leqslant s\leqslant -\frac{1}{2}, \\ -1 & \frac{1}{2}\leqslant s\leqslant 1, \end{cases}$$

$$|\varphi(s)|\leqslant 1;\ |\varphi'(s)|\leqslant C;\ |\varphi''(s)|\leqslant C.$$

We now split $\Phi(s)$ into the sum

$$\Phi(s)=\Phi_1(s)+\Phi_2(s),$$

where

$$\Phi_2(s)=\frac{1}{2}\Big[\Phi(-1)+\Phi(1)\Big]+\frac{1}{2}\Big[\Phi(-1)-\Phi(1)\Big]\varphi(s) \tag{III.16}$$

and

$$\Phi_1(s)=\Phi(s)-\Phi_2(s).$$

The function $\Phi_1(s)$ vanishes at the ends of the integration interval; $\Phi_2(s)$ is differentiable with respect to s. The function $\Phi_1(s, r, a)$ also vanishes for $r\to 0$, or more accurately, satisfies the estimate

$$|\Phi_1(s, r, a)|\leqslant Cr^{\mu}. \tag{III.17}$$

To see this, note that $d=\Phi(s, 0, a)$ does not depend on s or a, and is thus a constant. Adding and subtracting this constant in suitable places, we obtain

$$|\Phi_1(s, r, a)|\leqslant|\Phi(s, r, a)-d|+|\Phi(1, r, a)-d|+|\Phi(-1, r, a)-d|,$$

which in view of (III.2) implies (III.17).

We consider separately integrals of the type (III.15) with $\Phi_1(s)$ and $\Phi_2(s)$ in the integrand. In the case of $\Phi_2(s)$ it seems natural to integrate by parts

$$I_2(r, a, \xi)=\int_{-1}^{1}\Phi_2(s)\frac{dr}{r^2-2|a|rs+a^2-\xi}=\frac{1}{2|a|r}\int_{-1}^{1}\Phi_2'(s)d\ln(r^2-2|a|rs+a^2-\xi)=$$

$$=\frac{1}{2|a|r}\Bigg\{\Phi(-1)\ln[(r+|a|)^2-\xi]-\Phi(1)\ln[(r-|a|)^2-\xi]+$$

$$+\frac{1}{2}[\Phi(-1)-\Phi(1)]\int_{-1}^{1}\varphi'(s)\ln(r^2-2|a|rs+a^2-\xi)ds\Bigg\}. \tag{III.18}$$

Here we have chosen that branch of $\ln(-z)$ which is single-valued in the plane slit along the positive real axis and satisfies the condition

$$\operatorname{Im}\ln(-\omega^2+i0)=-\operatorname{Im}\ln(-\omega^2-i0).$$

The last integral on the right-hand side of (III.18) may be rewritten as

$$\frac{1}{4|a|r}[\Phi(-1)-\Phi(1)]\left[-\ln 2|a|r+\int_{-1}^{1}ds\,\varphi'(s)\ln(t-s)\right],$$

where $t=t(r, a, \xi)$ stands for the expression

$$t=\frac{r^2+a^2-\xi}{2|a|r}.$$

The integral with $\Phi_1(s)$ is also conveniently expressed in terms of $t(r, a, \xi)$:

$$I_1(r, a, \xi) = \int_{-1}^{1} \frac{\Phi_1(s)\,ds}{r^2 - 2|a|rs + a^2 - \xi} = \frac{1}{2|a|r} \int_{-1}^{1} \Phi_1(s) \frac{ds}{t - s}.$$

Our next problem, accordingly, is to examine the dependence of $t(r, a, \xi)$ on its variables.

Lemma III.3. The following estimates are valid

$$|t(r, a, \xi + \Delta) - t(r, a, \xi)| \leqslant C \frac{|\Delta|}{|a|r}; \qquad \text{(III.19)}_1$$

$$|t(r + \Delta, a, \xi) - t(r, a, \xi)| \leqslant CR \frac{|\Delta|}{|a|r^2}; \quad \Delta \leqslant \frac{1}{2} r; \qquad \text{(III.19)}_2$$

$$|t(r, a + h, \xi) - t(r, a, \xi)| \leqslant CR \frac{|h|}{r|a|\,|a + h|}; \qquad \text{(III.19)}_3$$

$$|t(r, a, \xi)| \leqslant \frac{CR}{|a|r}. \qquad \text{(III.19)}_4$$

Proof. We have

$$|t(r, a, \xi + \Delta) - t(r, a, \xi)| \leqslant \left| -\frac{\Delta}{2|a|r} \right| \leqslant C \frac{|\Delta|}{|a|r};$$

$$|t(r, a + h, \xi) - t(r, a, \xi)| \leqslant \left| \frac{a^2 + r^2 - \xi}{2|a + h|r} - \frac{a^2 + r^2 - \xi}{2|a|r} \right| +$$

$$+ \left| \frac{a^2 - (a + h)^2}{2|a + h|r} \right| \leqslant \frac{CR}{r} \frac{\big||a + h| - |a|\big|}{|a|\,|a + h|} \leqslant \frac{CR}{r} \frac{|h|}{|a|\,|a + h|};$$

$$|t| = \frac{|a^2 + r^2 - \xi|}{2|a|r} \leqslant \frac{CR}{|a|r};$$

the estimate (III.19)_2 is proved analogously to (III.19)_3, keeping in mind that $\frac{1}{r + \Delta} \leqslant \frac{2}{r}$ if $|\Delta| \leqslant \frac{1}{2} r$.

This completes the proof.

We now return to the integral (III.15).

Lemma III.4. The integral $I(r, a, \xi)$ may be represented as

$$I(r, a, \xi) = \frac{1}{2|a|r} \{2\pi f(-r\bar{a}) \ln[(r + |a|)^2 - \xi] +$$

$$+ 2\pi f(r\bar{a}) \ln[(r - |a|)^2 - \xi] + \mathcal{F}(r, a, \xi)\}, \qquad \text{(III.20)}$$

where $f(q)$ is the function appearing in the integrand in (III.4), and $\mathcal{F}(r, a, \xi)$ satisfies the estimates

$$|\mathcal{F}(r, a, \xi)| \leqslant CR^{\frac{\mu}{2}} \frac{r^{\frac{\mu}{2}}}{|a|^{\mu}}; \qquad \text{(III.21)}_1$$

$$|\mathcal{F}(r + \Delta, a, \xi) - \mathcal{F}(r, a, \xi)| \leqslant CR^{\frac{3\mu}{4}} \frac{|\Delta|^{\frac{\mu}{2}}}{|a|^{\frac{\mu}{2}}}; \qquad \text{(III.21)}_2$$

$$|\mathcal{F}(r, a + h, \xi) - \mathcal{F}(r, a, \xi)| \leqslant CR^{\frac{3\mu}{4}} \frac{|h|^{\frac{\mu}{2}}}{|a + \theta h|^{\mu}}; \qquad \text{(III.21)}_3$$

$$|\mathcal{F}(r, a, \xi + \Delta) - \mathcal{F}(r, a, \xi)| \leqslant CR^{\frac{\mu}{4}} \frac{|\Delta|^{\frac{\mu}{2}}}{|a|^{\frac{\mu}{2}}}. \qquad \text{(III.21)}_4$$

Proof. The first two terms in (III.20) come from the corresponding terms in (III.18). It is sufficient to observe that

$$\Phi(-1) = \int_0^{2\pi} d\varphi f(-\tilde{a}_1 r, -\tilde{a}_2 r, -\tilde{a}_3 r) = 2\pi f(-r\tilde{a})$$

and similarly

$$\Phi(1) = 2\pi f(r\tilde{a}).$$

All the remaining terms of $I(r, a, \xi)$ are gathered up in $\mathscr{P}(r, a, \xi)$:

$$\mathscr{P}(r, a, \xi) = \pi[f(r\tilde{a}) - f(-r\tilde{a})] \ln 2|a|r + \\ + \pi[f(-r\tilde{a}) - f(r\tilde{a})]\psi_1(t) + \psi_2(r, a, t), \tag{III.22}$$

where

$$\psi_1(t) = \int_{-1}^{1} \varphi'(s) \ln(t - s)\, ds \tag{III.23}$$

and

$$\psi_2(r, a, t) = \int_{-1}^{1} \frac{\Phi_1(s, r, a)}{t - s}\, ds. \tag{III.24}$$

The behavior of the functions $\psi_1(t)$ and $\psi_2(r, a, t)$ is easy to deduce. Thus we have the estimates

$$|\psi_1(t)| \leqslant C(1 + |t|)^\delta; \tag{III.25$_1$}$$

$$|\psi_1(t + \Delta) - \psi_1(t)| \leqslant C|\Delta|^\nu; \quad 0 \leqslant \nu \leqslant 1; \tag{III.25$_2$}$$

and

$$|\psi_2(r, a, t)| \leqslant Cr^\mu; \tag{III.26$_1$}$$

$$|\psi_2(r, a, t + \Delta) - \psi_2(r, a, t)| \leqslant Cr^\mu |\Delta|^{\frac{\mu}{2}}. \tag{III.26$_2$}$$

The estimate (III.25) follows for $|t| > 2$ from the simple estimate for $\ln z$

$$|\ln z| \leqslant C\left(|z| + \frac{1}{|z|}\right)^\delta, \tag{III.27}$$

where $\delta > 0$ is, as usual, an arbitrarily small number. When $|t| \leqslant 2$, we may once more integrate (III.23) by parts and make use of the boundedness of $\varphi''(s)$. The estimate (III.26) follows from (III.17), (III.10) and Privalov's lemma as stated in Appendix I.

Let us write down all the remaining estimates for the functions that occur in $\mathscr{P}(r, a, \xi)$. We estimate the Hölder differences with index $\frac{\mu}{2}$, since this index appears already in (III.26)$_2$. It follows from (III.27) and the formula for finite increments that

$$|\ln 2|a|r| = |\ln 2 + \ln|a| + \ln r| \leqslant CR^\delta\left(\frac{1}{|a|^\delta} + \frac{1}{r^\delta}\right); \tag{III.28$_1$}$$

$$|\ln 2|a|(r + \Delta) - \ln 2|a|r| \leqslant CR^\delta \frac{|\Delta|^{\frac{\mu}{2}}}{r^{\frac{\mu}{2} + \delta}}; \quad 0 \leqslant |\Delta| \leqslant \frac{1}{2} r; \tag{III.28$_2$}$$

$$|\ln 2|a + h|r - \ln 2|a|r| \leqslant CR^\delta \frac{|h|^{\frac{\mu}{2}}}{|a + \theta h|^{\frac{\mu}{2} + \delta}}. \tag{III.28$_3$}$$

Further, we find from (III.2) that

$$|f(r\tilde{a}) - f((r+\Delta)\tilde{a})| \leqslant C|\Delta|^{\mu}; \qquad (\mathrm{III}.29)_1$$

$$|f(r(\tilde{a}+h)) - f(r\tilde{a})| \leqslant Cr^{\frac{\mu}{2}} \frac{|h|^{\frac{\mu}{2}}}{|a|^{\frac{\mu}{2}}}. \qquad (\mathrm{III}.29)_2$$

To see how the second of these estimates is derived, note that by (III.2) we have

$$\left|f\left(r\frac{a+h}{|a+h|}\right) - f\left(r\frac{a}{|a|}\right)\right| \leqslant Cr^{\frac{\mu}{2}}\left|\frac{a+h}{|a+h|} - \frac{a}{|a|}\right|^{\frac{\mu}{2}}.$$

The last factor may be estimated as follows

$$\left|\frac{a+h}{|a+h|} - \frac{a}{|a|}\right| \leqslant \left|\frac{a}{|a|} - \frac{a+h}{|a|}\right| + |a+h|\left|\frac{1}{|a|} - \frac{1}{|a+h|}\right| \leqslant C\frac{|h|}{|a|},$$

which implies $(\mathrm{III}.29)_2$.

The function $f(r\tilde{a})$ occurs in (III.22) through

$$g(r, a) = f(r\tilde{a}) - f(-r\tilde{a}).$$

Applying (III.2) and (III.29), we obtain for $g(r, a)$ the estimates

$$|g(r, a)| \leqslant Cr^{\mu}; \qquad (\mathrm{III}.30)_1$$

$$|g(r+\Delta, a) - g(r, a)| \leqslant Cr^{\frac{\mu}{2}}|\Delta|^{\frac{\mu}{2}}; \quad |\Delta| \leqslant \frac{1}{2}r; \qquad (\mathrm{III}.30)_2$$

$$|g(r, a+h) - g(r, a)| \leqslant Cr^{\frac{\mu}{2}}\frac{|h|^{\frac{\mu}{2}}}{|a|^{\frac{\mu}{2}}}. \qquad (\mathrm{III}.30)_3$$

Finally, $\psi_2(r, a, t)$ is for fixed t a Hölder function of r and a with some index smaller than μ, and with the same estimating functions as $\Phi(s, r, a)$, whence

$$|\psi_{2.}(r+\Delta, a, t) - \psi_2(r, a, t)| \leqslant C|\Delta|^{\frac{\mu}{2}}; \qquad (\mathrm{III}.31)_1$$

$$|\psi_2(r, a+h, t) - \psi_2(r, a, t)| \leqslant CR^{\frac{\mu}{4}}\frac{|h|^{\frac{\mu}{2}}}{|a+\theta h|^{\frac{\mu}{2}}}. \qquad (\mathrm{III}.31)_2$$

The estimates (III.21) follow from the preceding estimates for all the functions entering in (III.22) and from the estimates (III.19). For example, the estimates $(\mathrm{III}.28)_1$, $(\mathrm{III}.30)_1$, $(\mathrm{III}.25)_1$, $(\mathrm{III}.19)_4$ and $(\mathrm{III}.26)_1$ give

$$|\mathscr{P}(r, a, \xi)| \leqslant CR^{\delta}\frac{r^{\mu-\delta}}{|a|^{\delta}}; \qquad (\mathrm{III}.32)$$

setting here $\delta = \frac{\mu}{2}$, we get $(\mathrm{III}.21)_1$. In order to derive $(\mathrm{III}.21)_2$, we consider the two cases:

1. $|\Delta| \leqslant \frac{1}{2}r$. We apply $(\mathrm{III}.28)_1$, $(\mathrm{III}.28)_2$, $(\mathrm{III}.30)_1$, $(\mathrm{III}.30)_2$, $(\mathrm{III}.25)_1$, $(\mathrm{III}.25)_2$, $(\mathrm{III}.31)_1$, $(\mathrm{III}.26)_2$, $(\mathrm{III}.19)_2$, and obtain

$$|\mathscr{P}(r+\Delta, a, \xi) - \mathscr{P}(r, a, \xi)| \leqslant CR^{\frac{\mu}{2}}\frac{|\Delta|^{\frac{\mu}{2}}}{|a|^{\frac{\mu}{2}}} + CR^{\frac{\mu}{2}}|\Delta|^{\frac{\mu}{2}} \leqslant CR^{\frac{3\mu}{4}}\frac{|\Delta|^{\frac{\mu}{2}}}{|a|^{\frac{\mu}{2}}}.$$

2. $|\Delta| \geqslant \frac{1}{2} r$. We separately estimate $\mathcal{S}(r+\Delta, a, \xi)$ and $\mathcal{S}(r, a, \xi)$ according to (III.32), set $\delta = \frac{\mu}{2}$, and obtain

$$|\mathcal{S}(r+\Delta, a, \xi) - \mathcal{S}(r, a, \xi)| \leqslant CR^{\frac{\mu}{2}} \frac{|r+\Delta|^{\frac{\mu}{2}} + r^{\frac{\mu}{2}}}{|a|^{\frac{\mu}{2}}} \leqslant CR^{\frac{\mu}{2}} \frac{|\Delta|^{\frac{\mu}{2}}}{|a|^{\frac{\mu}{2}}},$$

which proves $(\text{III}.21)_2$. The remaining estimates are derived analogously. This completes the proof.

It is characteristic of the estimates (III.21) that the estimating functions do not depend on r. It is therefore not difficult to estimate the integral

$$\mathcal{T}(a, \xi, \eta) = \int_0^\infty \frac{r dr}{r^2 - \eta} \mathcal{S}(r, a, \xi) \tag{III.33}$$

which upon substituting $r^2 = t$ assumes the form

$$\mathcal{T}(a, \xi, \eta) = \int_{-\infty}^{\infty} \frac{dt}{t-\eta} \mathscr{S}(t, a, \xi),$$

where

$$\mathscr{S}(t, a, \xi) = \begin{cases} \frac{1}{2} \mathcal{S}(\sqrt{t}, a, \xi), & t \geqslant 0, \\ 0, & t < 0, \end{cases}$$

is a Hölder function of t with index $\frac{\mu}{4}$, and vanishes for $|t| \geqslant CR$. Applying the lemma on singular integrals, we obtain for this integral the estimates of Lemma 6.2. The integral over r, containing the logarithmic terms of (III.20), may be converted to the form (6.23) by appropriately substituting $r' = -r$. It is not difficult to see that all the constants in the estimates of the type (III.21) may be taken proportional to $\|f\|_\mu$, where

$$\|f\|_\mu = \sup_{k, h} \left\{ |f(k)| + \frac{|f(k+h) - f(k)|}{|h|^\mu} \right\},$$

which explains the presence of this factor in the estimates of the function $\mathcal{T}$.

This ties up the proof of Lemma 6.2.

APPENDIX IV

Remarks and references

Introduction. We bring together here a few references to the literature on quantum mechanical scattering theory. The classical source for such references is the monograph by Mott and Massey /12/. The problems of scattering theory are treated in this book in the stationary formulation. The time-dependent formulation apparently has been first put forward by Møller /13/ in his papers, which constitute an extension of Heisenberg's studies in the theory of the scattering operator, or **S**-matrix. The technique developed by Møller was refined in the so-called formal theory of scattering. A typical statement of this formalism is given, e. g. , by Gell-Mann and Goldberger /14/. Ekstein /15/ specifically deals with the formal theory of scattering for systems consisting of three or more bodies. This paper also gives a procedure for the derivation of stationary solutions by means of a limiting process of transition from the complex plane in the kernel of the resolvent of the energy-operator.

§ 1. The variables κ, p in momentum space, used throughout this book, are conjugate to the so-called Jacobi coordinates, which are frequently used for the description of a three-body system in configuration representation.

Let us compare the conditions A_{θ_0} and B_{μ_0} which we impose on the potential $v(k)$ in momentum representation with the conditions usually imposed on the potential $\hat{v}(x)$ in configuration representation. A. Ya. Povzner /1/ assumes that $\hat{v}(x)$ is bounded and satisfies at infinity the estimate

$$|\hat{v}(x)| \leqslant C|x|^{-\alpha},$$

where $\alpha = 3.5+\varepsilon$, $\varepsilon > 0$. Ikebe /4/ reduces this to $\alpha = 2+\varepsilon$. It is possible to formulate sufficient conditions on $\hat{v}(x)$ to ensure that $v(k)$ satisfy conditions A_{θ_0} and B_{μ_0}, but these turn out to be far stronger than the above estimate. On the other hand, this estimate cannot be deduced from the conditions A_{θ_0} and B_{μ_0} alone. Thus, the conditions which we use in this work differ from those usually imposed on the potential in configuration representation. That the estimates derived in the main text are compatible with them indicates the suitability of these conditions for treatment in the momentum representation.

§ 3. A similar method to the one which we used for passing from equation (3.8) to the system of equations (3.13), is known in physical literature. Watson /16/ called it the "method of multiple scattering". The ideas connected with this treatment of the integral equations of scattering theory have recently found application in the solution of problems of statistical physics (cf., e. g., /17/). The use of a system of equations of the type (3.13) for a rigorous mathematical investigation of the resolvent of the energy operator of a three-body system has been put forward by the author /18, 19/. In /19/ there appears a proposition which is equivalent to Theorem 3.1.

§ 4. In our study of the integral equation (4.1) we first chose a Banach space such that equation (4.1) should become in it an equation of the second kind with a completely continuous operator, and showed that the corresponding homogeneous equation has a nontrivial solution only for those values of the parameter z which are discrete eigenvalues of the energy operator under consideration. A similar method was applied earlier by A. Ya. Povzner /1/ to the study of the integral equation for the kernel of the resolvent of an $\mathbf{h}$-type operator in configuration representation.

§§ 5, 9, 11. The results concerning the behavior of the resolvent of the energy operator for a three-body system and the proof of the expansion theorem for the eigenfunctions of this operator are due to the author and were first published in /20/.

The author has not succeeded in proving that the eigenfunctions of the discrete spectrum $\mathbf{H}$ are sufficiently smooth. Accordingly, it is an open question whether every such function corresponds to a solution of a homogeneous equation of the type

$$\mathbf{A}(z)\psi = 0$$

and whether all the eigenvalues of $\mathbf{H}$ have finite multiplicity.

We stress again that the assumption that the eigenvalues of the "two-body" $\mathbf{h}_\alpha$, $\alpha = 23, 31, 12$, are negative is essential for the foregoing analysis. It is this circumstance that enables us to separate so simply the singularities in the denominators.

§ 7. Lemma 7.8 is analogous to Lemma 2 of Povzner's paper /1/.

§ 10. Numerous mathematical papers deal with the existence of limits for $t \to \pm \infty$ for operators of the type

$$\exp\{i\mathbf{A}t\}\exp\{-i\mathbf{B}t\}$$

under various conditions on $\mathbf{A}$ and $\mathbf{B}$. The first of those seems to have been the work of Friedrichs /21/, subsequently continued by O. A. Ladyzhenskaya and the author /22/. Many references may be found in the review article by Kuroda /23/.

§ 11. There exist in the literature several definitions of the **S**-matrix for multi-channel scattering (cf. e. g. , /24/ and /25/). The definition adopted in this book was put forward by F. A. Berezin, R. A. Minlos and the author in a paper read at the Fourth All-Union Mathematical Conference held in Leningrad in June 1961, and is due to be published in the second volume of the proceedings of this conference.

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